

Corresponding editor: Hua Wang

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## A NEW KISSING NUMBER OF THE REGULAR POLYGON

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ABSTRACT. Let  $P_n$  be an arbitrary regular polygon with  $n$  sides. The kissing number  $k(P_n)$  is the maximum number of congruent regular polygons (copies of  $P_n$ ) that can be arranged so that each touches  $P_n$  but no two of them overlap. Youngs (1939), Klamkin (1995), Zhao (1998, 2002) and others established that  $k(P_3) = 12$ ,  $k(P_4) = 8$  and  $k(P_n) = 6$  with  $n \geq 5$ . In this paper, we will study a new kissing number. Let  $Q_n$  be a regular polygon with  $n$  sides whose length of side is the half of  $P_n$ . What is the maximum number  $k_1(P_n)$  of congruent regular polygons  $Q_n$  that can be arranged so that each touches  $P_n$  but no two of them overlap? We will prove that  $k_1(P_n) = 9$  when  $n \geq 9$ .

### 1. Introduction

Two plane figures are said to kiss each other if they do not overlap but their boundaries have nonempty intersection. For a plane figure  $F$ , a kissing configuration of order  $n$  consists of  $n + 1$  congruent nonoverlapping copies of  $F$  such that one copy kisses each of the remaining  $n$ . The kissing number of  $F$ , denoted by  $k(F)$ , is the largest integer  $n$  such that a kissing configuration of order  $n$  exists for  $F$ .

Let  $P_n$  be an arbitrary regular polygon with  $n$  sides. For the square  $P_4$ , Youngs ([1]) seems to have been the first to establish that  $k(P_3) = 12$  and  $k(P_4) = 8$ . The determination of  $k(P_4)$  was later posed as a Putnam competition problem. Youngs' result was reproduced in the solution book ([3]). Klamkin et al. ([2]) also gave an elementary proof that  $k(P_4) = 8$  and presented a few kissing configurations of order  $k(P_6) = 6$  for  $P_6$ . Zhao et al. ([5,6]) established that  $k(P_n) = 6$  with  $n \geq 5$ .

In this paper, we will study a new kissing number. Let  $Q_n$  be a regular polygon with  $n$  sides whose length of side is the half of  $P_n$ . What is the maximum number  $k_1(P_n)$  of congruent regular polygons  $Q_n$  that can be arranged so that each touches  $P_n$  but no two of them overlap? We will prove that  $k_1(P_n) = 9$  when  $n \geq 9$ .

### 2. Main result

**Theorem 2.1.** *Suppose that  $P_n$  is an arbitrary regular polygon with  $n$  sides and*

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2000 *Mathematics Subject Classification.* Primary 05C30.

*Key words and phrases.* Kissing number; regular polygon; distance.

\* This research is supported by natural scientific foundation of Shandong Province in China (ZR2009GM017).

$n \geq 9$ . Then

$$k_1(P_n) = 9.$$

In order to prove the above Theorem, we first prove the following Lemmas.

**Lemma 2.2.**  $k_1(P_n) \geq 9$ .

*Proof.* Suppose that  $P$  is an circle with radius 2, it is easy to see that there are 9 circles with radius 1 around  $P$  which are tangent to it. Since any regular polygon with  $n$  sides have an circumcircle, so the proposition holds.  $\square$

In the following, we always suppose that  $n \geq 9$ .

**Lemma 2.3.** Suppose that  $P_n$  is an arbitrary regular polygon with  $n$  sides. Let  $Q_n$  and  $Q'_n$  be two regular polygons with  $n$  sides whose length of side is the half of  $P_n$ .  $P_n$  kisses both  $Q_n$  and  $Q'_n$  such that  $Q_n$  and  $Q'_n$  kiss each other. Let  $O, A, B$  be the centers of  $P_n, Q_n$  and  $Q'_n$  respectively. Note  $\angle AOB$  by  $\theta_n$ . Then

$$\theta_n \geq \arccos \frac{8 - \cos \frac{2\pi}{n}}{9}.$$

*Proof.* Without loss of generality, we suppose that  $R = 2$ , then  $r_n = 2\cos \frac{\pi}{n}$ , where  $R$  is the radius of  $P_n$ ,  $r_n$  is the distance from the center of  $P_n$  to its side. Then we have  $R' = 1$  and  $r'_n = \cos \frac{\pi}{n}$ , where  $R'$  is the radius of  $Q_n$  and  $Q'_n$ ,  $r'_n$  is the distance from the centers of  $Q_n$  and  $Q'_n$  to their sides. Clearly, both  $OA$  and  $OB$  are at least  $r_n + r'_n = 3\cos \frac{\pi}{n}$  and at most  $R + R' = 3$ , and  $AB$  is at least  $2r'_n = 2\cos \frac{\pi}{n}$ .

Since  $AB \geq 2r'_n = 2\cos \frac{\pi}{n}$ , if we replace the triangle  $\triangle AOB$  by the triangle  $\triangle A_1OB_1$  with  $OA_1 = OA$ ,  $OB_1 = OB$ ,  $A_1B_1 = 2\cos \frac{\pi}{n}$ , then  $\angle AOB \geq \angle A_1OB_1$ . Let  $OA_1 = x$  and  $OB_1 = y$ . By the Law of Cosines, we have

$$\cos \angle A_1OB_1 = (x^2 + y^2 - 4\cos^2 \frac{\pi}{n})/2xy,$$

where  $3\cos \frac{\pi}{n} \leq x, y \leq 3$ . Let  $F(x, y) = \cos \angle A_1OB_1 = (x^2 + y^2 - 4\cos^2 \frac{\pi}{n})/2xy$ , then

$$F'_x(x, y) = (x^2 - y^2 + 4\cos^2 \frac{\pi}{n})/2x^2y,$$

$$F'_y(x, y) = (y^2 - x^2 + 4\cos^2 \frac{\pi}{n})/2y^2x.$$

Since  $n \geq 6$ , then  $\tan \frac{\pi}{n} \leq \tan \frac{\pi}{6} = \frac{\sqrt{3}}{3} < \frac{2}{3}$ . From  $3\cos \frac{\pi}{n} \leq x, y \leq 3$ , we have

$$x^2 - y^2 + 4\cos^2 \frac{\pi}{n} \geq (3\cos \frac{\pi}{n})^2 - 9 + 4\cos^2 \frac{\pi}{n} = 4\cos^2 \frac{\pi}{n} - 9\sin^2 \frac{\pi}{n} > 0.$$

This inequality implies that  $F'_x(x, y) > 0$  and  $F'_y(x, y) > 0$ . Therefore,

$$F(x, y) \leq F(3, y) \leq F(3, 3) = 1 - \frac{2}{9}\cos^2 \frac{\pi}{n} = \frac{8 - \cos \frac{2\pi}{n}}{9}.$$

Now, the above relation implies that

$$\theta_n = \angle AOB \geq \angle A_1OB_1 = \arccos F(x, y) \geq \arccos F(3, 3) \geq \arccos\left[\frac{8 - \cos\frac{2\pi}{n}}{9}\right].$$

This completes the proof.  $\square$

*Proof. Proof of the Theorem 2.1* Let  $\alpha_n = \arccos\left[\frac{8 - \cos\frac{2\pi}{n}}{9}\right]$ , then  $\theta_n \geq \alpha_n$ . Let  $\phi_n = \frac{360^\circ}{\theta_n}$ . When  $n = 9$ , we have  $\cos\frac{2\pi}{9} > 0.7660$ ,  $\alpha_9 > 36.50^\circ$ . So  $\theta_9 > 36.50^\circ$ . Thus

$$\phi_9 = \frac{360^\circ}{\theta_9} < \frac{360^\circ}{36.50^\circ} < 9.8631.$$

It follows that  $k_1(P_9) \leq 9$ .

When  $n \geq 9$ ,  $\cos\frac{2\pi}{n} < \cos\frac{2\pi}{n+1}$  implies

$$\arccos\frac{8 - \cos\frac{2\pi}{n}}{9} < \arccos\frac{8 - \cos\frac{2\pi}{n+1}}{9}.$$

It follows that  $\phi_n > \phi_{n+1}$ . When  $n \geq 9$  we have

$$k_1(P_n) \leq k_1(P_9) \leq 9.$$

From Lemma 2.1 it follows that  $k_1(P_n) = 9$  when  $n \geq 9$ .

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