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ON THE GENERALIZTION OF SOME THEOREMS ABOUT PSEUDORANDOM GRAPHS THROUGH AN EIGENVALUE

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ABSTRACT. We use a survey paper on pseudorandom graphs by Krivelevitch and Sudakov [1], presented some of the theorems on independent set, coloring and Hamiltoncity of pseudorandom in the form of proposition and generalized, proved them through an eigenvalue of graphs and also discussed sharp bounds for eigenvalues .

1. Introduction

Pseudo-random graphs are graphs which behave like random graphs. Random graphs have proven to be one of the most important and fruitful concepts in modern Combinatorics and Theoretical Computer Science. Pseudo-random graphs are modeled after truly random graphs, and therefore mastering the edge distribution in random graphs can provide the most useful insight on what can be expected from pseudo-random graphs. Many authors has been contributed a significant work on pseudorandom graphs (Viz., definitions, examples, properties, results and their applications in Mathematics and other fields). An interested reader can see the followings Andrew Thomason [2, 3], Chung, Graham and Wilson [4], Wormald [5] given in the reference. Next we recall some definitions.

Definition 1.1. Random graph : A random graph$G(n,p)$ is a probability space of all labeled graphs on n vertices 1, 2, . . .n probability of a graph G = (V, E) with n vertices and e edges in G(n p) is $p^e(1-p)^{\binom{n}{2}-e}$. We observe that for $p=\frac{1}{2}$ the probability of every graph is the same and for $p>\frac{1}{2}$ the probability of a graph G1 with more edges than another graph G2 is higher. (And the probability of G1 is smaller than the probability of G2 if $p<\frac{1}{2}$.) Almost all properties hold for all $G \in G(n,p)$ has property tends to one as n tends to infinity. The following definition and theorem are concerned about the edge distribution of random graphs.

Definition 1.2. Definition 1.1.2: (Random graph with given degree distribution): Given $w = w_1w_2 \dots w_n$ a sequence. A random graph with given degree distribution $G(w)$ is a probability space of all labeled graphs on n vertices 1, 2,. , n where $\{i,j\}$ is an edge of $G(w)$ with probability $w_iw_j\rho$, independently of any other edges, for $1 \le i < j \le n$. Where ρ plays the role of a normalization factor, i.e.$\rho = (\sum_{i=1}^{n} w_i)^{-1}$. .

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The expected degree of the vertex i is w_j. The above definition of a random graph with given degree distribution comes from Chung and Lu [6]. There are some other definitions for random graphs with given degrees (interested reader can see Watts, D. J., and et. all.,[7].

Next we generalize the concept of independent set, coloring, Hamiltoncity of pseudo-random graphs by means of the eigenvalues. Here we recall all the necessary prepositions are from [1] and generalized.

2. Independent set

An independent set in a graph is a set of vertices with no edges between them. The greatest integer r such that G contains an independent set of size is the independence number of G, and is denoted by $\alpha(G)$. Now we present a proposition (1.2.1) of [1] and generalize.

Proposition 2.1. *([1],p26) Let G be an (n, d, λ)-graph. Then*

$$\alpha(G) \leq \frac{\mu n}{d+\mu}.$$

Theorem 2.2. *Let G be an graph with second Laplacian eigenvalue λ and let U be an independent set. Then the independence number satisfy*

$$\mid U \mid \leq \frac{\lambda vol(G)}{\delta(1+\lambda)} \leq \frac{\lambda n \Delta}{\delta(1+\lambda)}.$$

Proof. We notice that the theorem follows from the following more general theorem by using the trivial bounds $vol(G) \leq \Delta n$ and $vol(G) \geq \delta|U|$. □

Theorem 2.3. *(Volume of independent set) Let G be a graph with second Laplacian eigenvalue λ and let U be an independent set. Then*

$$vol(U) \leq \frac{\lambda vol(G)}{1+\lambda}.$$

Proof. Because U is an independent set in G, it holds that e(U, U) = 0 and by Theorem 4.16 we have that

$$\frac{vol(U)^2}{vol(G)} \leq \frac{vol(U)vol(\bar{U})}{vol(G)} \leq \frac{vol(U)(vol(G)-vol(U))}{vol(G)}.$$

This implies that

$$vol(U) \leq \frac{\lambda vol(G)}{1+\lambda}.$$

□

Example 2.4. We look at the star S_n with n vertices. The second Laplacian eigenvalue of S_n is 1. We have also $\Delta = n-1$ and $\delta = 1$. So theorem 2.2 would give us that the size of any independent set in S_n is at most $n-1$, which is a sharp bound. We can conclude with Theorem 2.3 that the volume of an independent set in S_n is at most $n-1$. This bound is also sharp for S_n.

3. Coloring

A proper coloring of G is an assignment of colors to each vertex so that adjacent vertices receive different colors. The minimum number of colors required for that is the chromatic number of G. A perfect matching M in a graph G is a set of disjoint edges such that every vertex is incident to (exactly) one edge from M. Thus, a necessary condition for the existence of a perfect matching is that there is an even number of vertices.

Proposition 3.1. *([1], p.28) Let G be an (d, n, λ)-graph. Then the chromatic number satisfies*

$$\chi(G) \leq 1 + \frac{d}{\lambda}.$$

Theorem 3.2. *Let G be a graph on n vertices with second Laplacian eigenvalue Then the chromatic number satisfies*

$$\chi(G) \geq \frac{n\delta(1+\lambda)}{\lambda vol(G)} \geq \frac{\delta(1+\lambda)}{\lambda\Delta}.$$

Proof. Every color class in the proper coloring of G forms an independent set. By using theorem 2.2 we obtain:

$$\chi(G) \geq \frac{n}{\alpha(G)} \geq \frac{n\delta(1+\lambda)}{\lambda vol(G)} \geq \frac{\delta(1+\lambda)}{\lambda\Delta}.$$

□

Remark 3.3. Question: What are the possible values for the second Laplacian eigenvalue of graphs on n vertices which are k-colorable? Before going to answering this question, we need the following theorem which is special case of theorem 6.7 of Chungs book [8] and then we prove theorem 3.6.

Theorem 3.4 (8)**.** *Let G be a graph on n vertices with normalized Laplacian eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \ldots \leq \lambda_n$. If G is k-colorable, then $\lambda_n \geq \frac{k}{k-1}$.*

This bound is very sharp. Let us look at the k-partite graphs with parts of equal size. We can now compute the whole spectrum of the complete k-partite graph $K_{m,m,\ldots,m}$where $n = km$. We notice that $K_{m,m,\ldots,m}$ is the m- blow-up of the complete graph K_k. The normalized Laplacian spectrum of K_k is given by $spec(L(K_k) = 0, (\frac{k^{k-1}}{k-1})$. Therefore we have the largest normalized Laplacian eigenvalue is $\frac{k}{k-1}$.

Theorem 3.5. *Let G be a graph on n vertices with normalized Laplacian eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then the eigenvalues of the k-blow-up of G are 1 with multiplicity $n(k-1)$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$.*

Proof. We will only sketch a proof. The eigenvalues 1 are derived easily. There are $n(k-1)$ linearly independent eigenvectors of the form $(\ldots, 1, \ldots, -1, \ldots)$. Each eigenvalue of G is also an eigenvalue of G(k), since the eigenvector can be k- blown up. Finally we can check that we have enough eigenvalues $n(k-1) + n = nk$. □

Theorem 3.6. *i) The minimum value for the second Laplacian eigenvalue over all graphs on vertices which are k-colorable is* $\frac{1}{k-1}$ *for any* n*.*

ii) The maximum value for the second Laplacian eigenvalue over all graphs on n *vertices which are k-colorable is 1.*

Proof. i)By the theorem 3.4, every k-colorable graph has an eigenvalue which is at least $\frac{k}{k-1}$. So the second Laplacian eigenvalue of a k-colorable graph is at least $\frac{k}{k-1}$. We look at $K_k(s)$, i.e. the s-blow-up of the complete graph K_k . Then theorem 3.5 give us that the second Laplacian eigenvalue of $K_k(s)$ is $\frac{k}{k-1}$. So this proves the first the equality.

ii)We distinguish two cases.

Case 1: assume that $k = 1$, then the only 1-colorable graphs are union of points and they have second Laplacian eigenvalue 1.

Case 2: For $k \geq 2$, we can take a bipartite graph which is then k-colorable and has second Laplacian eigenvalue 1.

□

4. Hamiltoncity

A Hamiltonian cycle of a graph G = (V, E) is cycle of length $n = |V|$, i.e. the cycle goes through all vertices once. A graph is called Hamiltonian if it consists a Hamiltonian cycle. We will use the following theorem by Chvatal and Erdos.

Proposition 4.1. *([9]) Let G be a graph with at least three vertices. If, for some s, G is s-connected and contains no independent set of size more than s, then G is Hamiltonian.*

Now, we can get a condition for the Hamiltoncity of a graph by combining this proposition with proposition (1.2.1) of [10] and above proposition 2.1.

Proposition 4.2. *([1], p.37) Let G be an* (d, n, λ) *-graph. If*

$$d - 36\frac{\lambda^2}{d} \geq \frac{\lambda n}{d + \lambda},$$

then G is Hamiltonian.

In the same way, by combining Proposition 4.2 with proposition 2.1 and Theorem (1.2.2) of [10] we get the following result.

Theorem 4.3. *(Hamiltoncity)*

Let G be a graph on n vertices with second Laplacian eigenvalue λ*.*

If

$$\delta - 36\lambda^2\Delta \geq \frac{n\delta(1 + \lambda)}{\lambda vol(G)}$$

, then G is Hamiltonian.

5. Conclusion

We discussed some theorems on independent set, coloring and Hamiltoncity of pseudorandom in the form of proposition and generalized, proved them through an eigenvalue of graphs. We also discussed sharp bounds for eigenvalues of graphs.

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