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SEVERAL NEW COMBINATORIAL IDENTITIES ON THE STIRLING NUMBER

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ABSTRACT. A box contains m balls of m distinct colors, the same size and shape. Draw balls successively n times at random independently and with replacement from the box. Find the the expectation of the number of the drawn colors. Some combinatorial identities on the Stirling number of the second kind $S(n,m)$ are derived based on the solution to the problem by using probabilistic method.

1. Introduction

It is an important method to find and prove some combinatorial identities based on a probabilistic model[1,2]. In this paper, We build a model as follows:

A box contains m balls of the same size and shape but of m distinct colors. Draw balls successively n trials at random and with replacement from the box, which is noted as test T. Investigate the the number of the drawn different colors.

2. Distribution

Let X be the number of the different balls in test T, $X = 1, 2, \cdots, m$. The events A_i represents the event "the i-th kind ball is drawn", $i = 1, 2, \cdots, m$. $P(A_i)$ denotes the probability of A_i.

$$\begin{aligned} & P(A_1A_2\cdots A_k) = 1 - P(\bar{A}_1 \cup \bar{A}_2 \cup \cdots \cup \bar{A}_k) \\ = \quad & 1 - \binom{k}{1}P(\bar{A}_1) + \binom{k}{2}P(\bar{A}_1\bar{A}_2) - \cdots + (-1)^k\binom{k}{k}P(\bar{A}_1\bar{A}_2\cdots\bar{A}_k) \\ = \quad & 1 - \binom{k}{1}\Big(\frac{m-1}{m}\Big)^n + \binom{k}{2}\Big(\frac{m-2}{m}\Big)^n - \cdots + (-1)^k\binom{k}{k}\Big(\frac{m-k}{m}\Big)^n. \end{aligned}$$

thus,

$$P(A_1A_2\cdots A_k) = \sum_{k=0}^{k}(-1)^i\binom{k}{i}\Big(\frac{m-i}{m}\Big)^n.$$

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Note that

$$\begin{aligned} & P\Big(\bigcup_{l=k+1}^{m}(A_1A_2\cdots A_kA_l)\Big) \\ = & \sum_{i=1}^{m-k}(-1)^{i-1}\binom{m-k}{i}P(A_1A_2\cdots A_kA_{k+1}\cdots A_{k+i}) \\ = & \sum_{i=1}^{m-k}(-1)^{i-1}\binom{m-k}{i}P(A_1A_2\cdots A_{k+i}). \end{aligned}$$

so

$$\begin{aligned} P\{X=k\} &= \binom{m}{k}P(A_1A_2\cdots A_k\bar{A}_{k+1}\bar{A}_{k+2}\cdots\bar{A}_m) \\ &= \binom{m}{k}\Big[P(A_1A_2\cdots A_k)-P\Big(\bigcup_{l=k+1}^{m}(A_1A_2\cdots A_kA_l)\Big)\Big] \\ &= \binom{m}{k}\sum_{i=0}^{m-k}(-1)^i\binom{m-k}{i}P(A_1A_2\cdots A_{k+i}) \\ &= \binom{m}{k}\sum_{i=0}^{m-k}(-1)^i\binom{m-k}{i}\sum_{j=0}^{k+i}(-1)^j\binom{k+i}{j}\Big(\frac{m-j}{m}\Big)^n. \end{aligned}$$

Especially, the probability of the all kinds of balls are drawn out in n times is

$$P\{X=m\}=\sum_{i=0}^{m}(-1)^i\binom{m}{i}\Big(\frac{m-i}{m}\Big)^n=\frac{m!S(n,m)}{m^n}. \tag{1}$$

where $n\geq m$, and $S(n,m)$ is the Stirling number of the second kind [3].
For (1), it is corresponding to n distinct balls are set into $m(n\geq m)$ distinct boxes with no vacant box, then the number of approaches is $m!S(n,m)$. It is equivalent to a set with n elements is divided into m vacant subsets.

Thus, by the definition of $S(n,k)$, we have

$$P\{X=k\}=\frac{A_m^kS(n,k)}{m^n},\ \ k=1,2,\cdots,\min(n,m).$$

Then, we obtain a combinatorial identity

$$\sum_{k=1}^{\min(n,m)}A_m^kS(n,k)=m^n. \tag{2}$$

where $A_m^k=m(m-1)\cdots(m-k+1)$.

Because $\sum\limits_{k=1}^{m}P\{X=k\}=1$, we have

$$\sum_{k=1}^{m}\sum_{i=0}^{m-k}\sum_{j=0}^{k+i}(-1)^{i+j}\binom{m}{k}\binom{m-k}{i}\binom{k+i}{j}\Big(\frac{m-j}{m}\Big)^n=1.$$

i.e.

$$m^n = \sum_{k=1}^{m}\sum_{i=0}^{m-k}\sum_{j=0}^{k+i}(-1)^{i+j}\binom{m}{k}\binom{m-k}{i}\binom{k+i}{j}(m-j)^n. \qquad (3)$$

At the same time, by the distribution of X using different methods as above, a new explicit expression on $S(n,k)$ is derived:

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{m-k}\sum_{j=0}^{k+i}(-1)^{i+j}\binom{m-k}{i}\binom{k+i}{j}(m-j)^n. \qquad (4)$$

3. Expectation

Set

$$X_i = \begin{cases} 1, & \text{the i-th kind color is drawn out in n trials}, \\ 0, & otherwise. \end{cases}$$

Then

$$X = X_1 + X_2 + \cdots + X_m.$$

Because $P\{X_i = 0\} = \left(\frac{m-1}{m}\right)^n$, then the mathematical expectation $EX = m\Big(1 - \left(\frac{m-1}{m}\right)^n\Big)$. By $EX = \sum_{k=1}^{m} kP\{X = k\}$, we obtain

$$\sum_{k=1}^{m}\sum_{i=0}^{m-k}\sum_{j=0}^{k+i}(-1)^{i+j}\binom{m}{k}\binom{m-k}{i}\binom{k+i}{j}\Big(\frac{m-j}{m}\Big)^n k = m\Big(1-\big(\frac{m-1}{m}\big)^n\Big). \qquad (5)$$

$$\sum_{k=1}^{\min(n,m)} kA_m^k S(n,k) = m^{n+1} - m(m-1)^n. \qquad (6)$$

4. Further Investigation

Suppose a box contains m distinct color balls. Draw ball out from the box at random one by one with replacement until all colors are drawn. Suppose we need at least Y times to draw the all colors, then $Y = m, m+1, \cdots$. Set $p_n = P\{X = n\}, q_n = P\{Y = n\}$, then

$$p_n = q_m + q_{m+1} + \cdots + q_{n-1} + q_n.$$

That is

$$\begin{aligned} q_n &= p_n - p_{n-1} \\ &= \frac{m!S(n,m)}{m^n} - \frac{m!S(n-1,m)}{m^{n-1}} = \frac{(m-1)!S(n-1,m-1)}{m^{n-1}} \\ &= \sum_{k=0}^{m-1}(-1)^k\binom{m-1}{k}\Big(\frac{m-1-k}{m}\Big)^{n-1}. \end{aligned}$$

We obtains an interesting formula on $S(n,m)$

$$\sum_{n=m}^{\infty}\frac{m!S(n,m)}{(m+1)^n}=1. \tag{7}$$

Specially, when $m=1,3,5$ in (7) respectively, it is

$$\sum_{n=2}^{\infty}\frac{1}{2^{n-1}}=1;\ \sum_{n=3}^{\infty}\frac{2^{n-2}-1}{3^{n-1}}=\frac{1}{2};\ \sum_{n=5}^{\infty}\frac{S(n,5)}{6^n}=\frac{1}{5!}. \tag{8}$$

In addition,

$$EY=\sum_{n=m}^{\infty}\frac{(m-1)!nS(n-1,m-1)}{m^{n-1}}.$$

We need draw Z_i times at random till the i-th kind color appears after the $(i-1)$-th kind color appears, $i=1,2,\cdots,m$(note that the probability $P\{Z_1=1\}=1$). Apparently, random variables $Z_1,Z_2,\cdots,Z_m$ independent each other, and

$$Y=Z_1+Z_2+Z_3+\cdots+Z_m=1+Z_2+Z_3+\cdots+Z_m.$$

$$P\{Z_k=i\}=\Big(\frac{k-1}{m}\Big)^{i-1}\Big(1-\frac{k-1}{m}\Big),\ \ i=1,2,\cdots;\ \ k=1,2,\cdots,m.$$

Z_k have the hypergeometric distribution, and the expectation $E(Z_k)=\frac{m}{m-k+1}$, $k=1,2,\cdots,m$, thus

$$EY=m\Big(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\Big)=mh_1(m).$$

where $h_1(m)=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}$ is the harmonic number.

Thinking of the above expectation $E(Y)$ just obtained,

$$\sum_{n=m}^{\infty}\frac{(m-1)!nS(n-1,m-1)}{m^{n-1}}=m\Big(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\Big). \tag{9}$$

By the independence of $Z_1,Z_2,\cdots,Z_m$ and the property of variance,

$$D(Z_k)=\frac{m(k-1)}{(m-k+1)^2},k=1,2,\cdots,m;$$

$$E(Y^2)=DY+(EY)^2=m^2(h_1^2(m)+h_2(m))-mh_1(m),$$

then

$$\sum_{n=m}^{\infty}\frac{(m-1)!n^2S(n-1,m-1)}{m^{n-1}}=m^2(h_1^2(m)+h_2(m))-mh_1(m). \tag{10}$$

where $h_2(m)=1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{m^2}$.

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