Journal of Combinatorics, Graph Theory and Applications
Vol. 1 No. 2 (July-December, 2026)

Received: 20th September 2025
Revised: 10th December 2025
Accepted: 06th January 2026

Corresponding editor: TianMing Wang

INDEPENDENCE POLYNOMIALS $I_\alpha(G;x)$ AND MERRIFIELD-SIMMONS INDEXES

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Abstract. In graph theory,independence polynomial $I_\alpha(G;x)$ is NP-hard (see [1] and [2]).In this paper, our ways are combinatorial counting methods. By means of the complementary of any stable set is a complete graph, conversely,the complementary of any complete graph is a stable set,the authors derive independence polynomial $I_\alpha(G;x)$,and present the explicit formulas of independence polynomials $I_\alpha(G;x)$ for a great deal of graphs,specially, independence polynomials of complete d-partite graph and (n-3)-regular graph. Finally, Merrifield-Simmons indexes are given.

1. Introduction

In this paper, the authors will solve Independence polynomials $I_\alpha(G;x)$ and Merrifield-Simmons indexes.

Definition 1.1. In graph theory, stable set (or independent set) is the set of vertices in a graph , no two of which are adjacent. That is, it is a set S of vertices such that for every two vertices in S, there is no edge connecting the two.

Definition 1.2. Independent set polynomials $I(G;x)$ are defined as

$$I(G;x)=\sum_{k=1}^{n} b_k(G)x^k=\sum_{I\subset v(G)}\prod_{v\in I}x,$$

where let $b_k(G)$ be exactly k-independent sets of G.

Complexity: It is easy to see that $I(G;x)$ is NP-hard to compute. (see [1])

Definition 1.3. If s_k denotes the number of stable sets of cardinality k in graph G,and $\alpha(G)$ is the size of a maximum stable set,then

$$I_\alpha(G;x)=\sum_{k=1}^{\alpha(G)} s_k x^k$$

is called independence polynomial of G.

(Also see[2])

2000 *Mathematics Subject Classification.* Primary 05A18; Secondary 05C10.

Key words and phrases. stable set; independence polynomial; Merrifield-Simmons index;

* This research is supported by Development Foud of Dali University(KY231901540).

Definition 1.4. For Merrifield-Simmons index of the graph, denoted by i(G), Merrifield-Simmons proposed, it is defined as the number of all independent sets of the graph, including the empty set.

$S(G)$ denotes the number of all stable sets in G, namely,

$$S(G)=\sum_{k=1}^{\alpha(G)} s_k.$$

Remark:(reviewing the size of maximum independent set) Because it is NP-hard that $\alpha(G)$ is the size of a maximum stable set (the size of maximum independent set), so far there exists not the explicit formula, a number of mathematicians have studied that $\alpha(G)$ is the size of a maximum stable set(the size of maximum independent set). Enumeration of $S(G)$ is more difficult than $\alpha(G)$.

2. Basic Lemmas

In the section, the authors will state some basic Lemmas used in the article.

Lemma 2.1. *If s_k denotes the number of stable sets of cardinality k in graph G, and c_k denotes the number of complete subgraphs of order k in $\bar{G}$, then $s_k = c_k$.*

Proof. Our idea is that two sets have the same cardinal number if there exists a 1-1 correspondence between them. Because the complementary of any stable set is a complete graph, conversely, the complementary of any complete graph is a stable set, then there exists a 1-1 correspondence between stable sets of cardinality k in G and complete subgraphs of order k in $\bar{G}$, so $s_k = c_k$. □

Lemma 2.2. *If $\alpha(G)$ is the size of a maximum stable set in G, then $\alpha(G)$ is the size of a maximum complete graph in $\bar{G}$.*

Proof. Omitted. □

3. Main resluts

In the section, the authors will discuss independence polynomials $I_\alpha(G;x)$ of graphs.

Theorem 3.1. *If c_k denotes the number of complete subgraphs of order k in $\bar{G}$, then independence polynomials $I_\alpha(G;x)$ of graphs are as follows:*

$$I_\alpha(G;x)=\sum_{k=1}^{\alpha(G)} c_k x^k,$$

where $\alpha(G)$ is the size of a maximum complete graph in $\bar{G}$.

Proof. By definition 1.3 , Lemma 2.1 and Lemma 2.2, then

$$\begin{aligned} I_\alpha(G;x) &= \sum_{k=1}^{\alpha(G)} s_k x^k \\ &= \sum_{k=1}^{\alpha(G)} c_k x^k, \end{aligned}$$

where $\alpha(G)$ is the size of a maximum complete graph in $\bar{G}$. □

Theorem 3.2. *If G is a graph with n vertices, the degree sequence of G is $d_1, d_2, \cdots, d_n$, $\bar{G}$ is no K_3 subgraph, then*

$$I_\alpha(G;x) = nx + [\binom{n}{2} - \frac{d_1 + d_2 + \cdots + d_n}{2}]x^2.$$

Proof. Because $\bar{G}$ is no K_3 subgraph, a maximum complete graph in $\bar{G}$ is K_2, then $\alpha(G) = 2$.

$c_k =$ the number of complete subgraphs of order k in $\bar{G}$,

$c_1 = n$

the number of edges of $G = \frac{d_1+d_2+\cdots+d_n}{2}$,

the number of edges of $\bar{G} = \binom{n}{2} - \frac{d_1+d_2+\cdots+d_n}{2}$,

then $c_2 = \binom{n}{2} - \frac{d_1+d_2+\cdots+d_n}{2}$.

Finally, according to Theorem 3.1, the result is derived as follows:

$$I_\alpha(G;x) = nx + [\binom{n}{2} - \frac{d_1 + d_2 + \cdots + d_n}{2}]x^2.$$

□

Theorem 3.3. *If G is a complete d-partite graph $K_{n,n,\cdots,n}$, the number of n is d, then*

$$I_\alpha(G;x) = d(1+x)^n - d.$$

Proof. Suppose that G is a complete d-partite graph $K_{n,n,\cdots,n}$, the number of n is d, then $\bar{G} = K_n \bigcup K_n \bigcup \cdots \bigcup K_n$, and the number of K_n is d.

c_1 is the number of complete subgraphs of order 1 in $\bar{G}$, $c_1 = d\binom{n}{1}$;

c_2 is the number of complete subgraphs of order 2 in $\bar{G}$, $c_2 = d\binom{n}{2}$;

......

c_k is the number of complete subgraphs of order k in $\bar{G}$, $c_k = d\binom{n}{k}$;

......

c_n is the number of complete subgraphs of order n in $\bar{G}$, $c_n = d\binom{n}{n}$;

$\alpha(G) = n$

Finally, according to Theorem 3.1, then

$$\begin{aligned} I_\alpha(G;x) &= d\binom{n}{1}x + d\binom{n}{2}x^2 + \ldots\ldots + d\binom{n}{k}x^k + \ldots\ldots + d\binom{n}{n}x^n \\ &= d[1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots\ldots + \binom{n}{k}x^k + \ldots\ldots + \binom{n}{n}x^n] - d \\ &= d(1+x)^n - d. \end{aligned}$$

□

Theorem 3.4. *If $G = K_{1,n-1}$ is a star graph with n vertices, then*

$$I_\alpha(G;x) = (x-1) + (1+x)^{n-1}.$$

Proof. Because of $G = K_{1,n-1}$, $\bar{G} = K_1 \bigcup K_{n-1}$ then $c_1 = n$;$c_2 = \binom{n-1}{2}$;......;$c_k = \binom{n-1}{k}$; ......;$c_{n-1} = \binom{n-1}{n-1}$. $\alpha(G) = n-1$. By Theorem 3.1 then

$$I_\alpha(G;x) = nx + \binom{n-1}{2}x^2 + ... + \binom{n-1}{k}x^k + ... + \binom{n-1}{n-1}x^{n-1}$$

$$= (x-1) + 1 + \binom{n-1}{1}x + \binom{n-1}{2}x^2 + ... + \binom{n-1}{k}x^k + ... + \binom{n-1}{n-1}x^{n-1}$$

$$I_\alpha(G;x) = (x-1) + (1+x)^{n-1}.$$

□

Theorem 3.5. *If G is a (n-2)-regular graph with n (even 2m) vertices, then independence polynomial of G is as follows:*

$$I_\alpha(G;x) = nx + \frac{n}{2}x^2.$$

Proof. Let G be a (n-2)-regular graph with n (even 2m), then $\bar{G}$ is a 1-regular graph,namely, $\bar{G} = K_2 \bigcup K_2 \bigcup \cdots \bigcup K_2$, and the number of K_2 is m.Because $\bar{G}$ is no K_3 subgraph, according to Theorem 3.2,

$$I_\alpha(G;x) = nx + [\binom{n}{2} - \frac{d_1 + d_2 + \cdots + d_n}{2}]x^2,$$

$$d_1 + d_2 + \cdots + d_n = n(n-2)$$

$$I_\alpha(G;x) = nx + \frac{n}{2}x^2.$$

□

Theorem 3.6. *If G is a n-3-regular graph with n vertices, $n \geq 6$, and $\bar{G} \cong C_n$, then independence polynomial of G is as follows:*

$$I_\alpha(G;x) = nx + nx^2.$$

Proof. Let G be a n-3-regular graph with n vertices, $n \geq 6$ and $\bar{G} \cong C_n$, because $\bar{G}$ is a 2-regular graph, the graph would be able to join the disjoint cycles, thus assume that C_n, say. Because $\bar{G}$ is no K_3 subgraph,according to Theorem 3.2,

$$I_\alpha(G;x) = nx + [\binom{n}{2} - \frac{d_1 + d_2 + \cdots + d_n}{2}]x^2,$$

$$d_1 + d_2 + \cdots + d_n = n(n-3)$$

$$I_\alpha(G;x) = nx + nx^2.$$

□

Corollary 3.7. *If G is a (n-3)-regular graph with n vertices, and*

$$\bar{G} = C_{n_1} \bigcup C_{n_2} \bigcup \cdots \bigcup C_{n_q},$$

$n_1 + n_2 + \cdots + n_q = n$, $C_{n_i} \cap C_{n_j} = \phi$ *for any i and j,* $i \neq j$, $3 \leq n_j \leq n, 1 \leq j \leq q, q \geq 1$, $n \geq 6$, *the number of* $n_j = 3$ *is* l, *then independence polynomial of G is given as follows:*

$I_\alpha(G;x) = nx + nx^2 + lx^3.$

Proof. $c_k =$ the number of complete subgraphs of order k in $\bar{G}$,
$\bar{G} = C_{n_1} \bigcup C_{n_2} \bigcup \cdots \bigcup C_{n_q}$, $n_1 + n_2 + \cdots + n_q = n$, $C_{n_i} \cap C_{n_j} = \phi$ for any i and j, $i \neq j$, $3 \leq n_j \leq n, 1 \leq j \leq q, q \geq 1$, $n \geq 6$,the number of $n_j = 3$ is l,
so $c_1 = n$;$c_2 = n$;$c_3 = l$; $\alpha(G) = 3$. According to Theorem 3.1,

$$I_\alpha(G;x) = \sum_{k=1}^{\alpha(G)} c_k x^k$$
$$= nx + nx^2 + lx^3.$$

□

Theorem 3.8. *If T is a tree with n vertices, then*
$I_\alpha(\bar{T};x) = nx + (n-1)x^2$.

Proof. Suppose that T is a tree with n vertices, the degree sequence of T is $d_1, d_2, \cdots, d_n$, then T is no K_3 subgraph, and the degree sequence of $\bar{T}$ is $n - 1 - d_1, n - 1 - d_2, \cdots, n - 1 - d_n$. According to Theorem 3.2,

$$I_\alpha(\bar{T};x) = nx + [\binom{n}{2} - \frac{(n-1-d_1) + (n-1-d_2) + \cdots + (n-1-d_n)}{2}]x^2$$
$$= nx + \frac{d_1 + d_2 + \cdots + d_n}{2} x^2$$
$$d_1 + d_2 + \cdots + d_n = 2\varepsilon = 2(n-1)$$

$$I_\alpha(\bar{T};x) = nx + (n-1)x^2.$$

□

4. Merrifield-Simmons index

In the setion, the authors will discuss Merrifield-Simmons indexes of graphs.

Theorem 4.1. *If $S(G)$ denotes the number of all stable sets of a graph in G, namely,*

$$S(G) = \sum_{k=1}^{\alpha(G)} s_k,$$

then

$$S(G) = \sum_{k=1}^{\alpha(G)} c_k,$$

where $\alpha(G)$ is the size of a maximum complete graph in $\bar{G}$.

Proof. By Lemma 2.1, $s_k = c_k$, By Lemma 2.2, $\alpha(G)$ is the size of a maximum stable set in G, $\alpha(G)$ is the size of a maximum complete graph in $\bar{G}$, so

$$S(G) = \sum_{k=1}^{\alpha(G)} s_k$$
$$= \sum_{k=1}^{\alpha(G)} c_k.$$

□

Theorem 4.2. *If $S(G)$ denotes the number of all stable sets of a graph in G, then*

$$S(G) = I_\alpha(G;1).$$

Proof.

$$S(G) = \sum_{k=1}^{\alpha(G)} c_k.$$

By Theorem 3.1,

$$I_\alpha(G;x) = \sum_{k=1}^{\alpha(G)} c_k x^k.$$

Let $x = 1$.Then

$$S(G) = I_\alpha(G;1).$$

□

Theorem 4.3. *If G is any graph, then the relation between independencs polynomial and Merrifield-Simmons index is as follows:*

$$i(G) = I_\alpha(G;1) + 1.$$

Proof.

$$i(G) = S(G) + 1 = I_\alpha(G;1) + 1.$$

□

Theorem 4.4. *Suppose that G is a complete d-partite graph $K_{n,n,\cdots,n}$, the number of n is d, then*

$$i(G) = d2^n - d + 1.$$

Proof. Method 1 Because of

$$\begin{aligned} S(G) = \sum_{k=1}^{\alpha(G)} c_k &= \sum_{k=1}^{n} d\binom{n}{k} \\ &= d\sum_{k=1}^{n}\binom{n}{k} \\ &= d\sum_{k=0}^{n}\binom{n}{k} - d \\ &= d2^n - d, \end{aligned}$$

then

$$i(G) = S(G) + 1 = d2^n - d + 1.$$

Method 2

By Theorem 3.3,

$$I_\alpha(G;x) = d(1+x)^n - d.$$

By Theorem 4.3,

$$i(G) = I_\alpha(G;1) + 1 = d2^n - d + 1.$$

□

Theorem 4.5. *If* $G = K_{1,n-1}$ *is a star graph with n vertices, then*

$$i(G) = 2^{n-1} + 1.$$

Proof. Method 1 For

$$\begin{aligned} S(G) &= \sum_{k=1}^{\alpha(G)} c_k \\ &= n + \binom{n-1}{2} + ... + \binom{n-1}{k} + ... + \binom{n-1}{n-1} \\ &= 1 + \binom{n-1}{1} + \binom{n-1}{2} + ... + \binom{n-1}{k} + ... + \binom{n-1}{n-1} \\ &= 2^{n-1}, \end{aligned}$$

then

$$i(G) = S(G) + 1 = 2^{n-1} + 1.$$

Method 2

By Theorem 3.4,

$$I_\alpha(G;x) = (x-1) + (1+x)^{n-1}.$$

By Theorem 4.3,

$$i(G) = I_\alpha(G;1) + 1 = 2^{n-1} + 1.$$

□

Theorem 4.6. *If G is a (n-2)-regular graph with n (even 2m) vertices, then*

$$i(G) = \frac{3n}{2} + 1.$$

Proof. Omitted. □

Theorem 4.7. *If G is a n-3-regular graph with n vertices,* $n \geq 6$*, and* $\bar{G} \cong C_n$*, then*

$$i(G) = 2n + 1.$$

Proof. Omitted. □

Corollary 4.8. *If G is a (n-3)-regular graph with n vertices, and*

$$\bar{G} = C_{n_1} \bigcup C_{n_2} \bigcup \cdots \bigcup C_{n_q},$$

$n_1 + n_2 + \cdots + n_q = n$, $C_{n_i} \cap C_{n_j} = \phi$ *for any i and j,* $i \neq j$, $3 \leq n_j \leq n, 1 \leq j \leq q, q \geq 1$, $n \geq 6$, *the number of* $n_j = 3$ *is l, then*

$$i(G) = 2n + l + 1.$$

Proof. By Corollary 3.7,

$$I_\alpha(G;x) = nx + nx^2 + lx^3.$$

By Theorem 4.3,

$$i(G) = I_\alpha(G;1) + 1 = n + n + l + 1 = 2n + l + 1.$$

□

Theorem 4.9. *If T is a tree with n vertices, then*

$$i(\bar{T}) = 2n.$$

Proof. By Theorem 3.8,

$$I_\alpha(G;x) = nx + (n-1)x^2.$$

By Theorem 4.3,

$$i(G) = I_\alpha(G;1) + 1 = n + (n-1) + 1 = 2n - 1 + 1 = 2n.$$

□

Theorem 4.10. *If G is a graph with n vertices, the degree sequence of G is $d_1, d_2, \cdots, d_n$, $\bar{G}$ is no K_3 subgraph, then*

$$i(G) = n + \binom{n}{2} - \frac{d_1 + d_2 + \cdots + d_n}{2} + 1.$$

Proof. By Theorem 3.2,

$$I_\alpha(G;x) = nx + [\binom{n}{2} - \frac{d_1 + d_2 + \cdots + d_n}{2}]x^2.$$

By Theorem 4.3,

$$i(G) = I_\alpha(G;1) + 1 = n + \binom{n}{2} - \frac{d_1 + d_2 + \cdots + d_n}{2} + 1.$$

□

Acknowledgments The authors are indebted to Professor Tianming Wang and the referees for their time and comments.

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