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THE REPRESENTING FORMULA OF N(G,K)

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ABSTRACT. In this paper, by combinatorial methods, the authors derive the representing formula between the number $N(G,k)$ of $S^{(n)} = \{K_i : 1 \le i \le n\}$-factors with exactly k components and coefficients of the chromatic polynomial of $\bar{G}$, and the representing formula between the number $A(G)$ of all $S^{(n)}$-factors and coefficients of the chromatic polynomial of $\bar{G}$, present classes of graphs $N(G,k)$ and $A(G)$, and give combinatorial identities related to $N(G,k)$.Finally, we solve the counting formula of K_d-factors in any complete d-partite graphs,the counting theorem of $K_l - factors$ in any graph G without any K_{l+1} subgraph, the counting formula of the number of covers on shortest circles, as well as the explicit formula $\mu(K_n^d)$ of the mean color numbers of K_n^d.

1. Introduction

In this paper, we propose the concept of $S^{(n)} = \{K_i : 1 \le i \le n\}$-factor. Recently, there are a number of necessary and sufficient conditions of 1-factor or (g,f)-factor. When $S^{(n)} = \{K_i : 2 \le i \le 2\}$(the number of K_2 is m),1-factor (or perfect matching) is a special case of $S^{(n)} = \{K_i : 1 \le i \le n\}$-factor. $S^{(n)}$-factors are different from (g,f)-factors, contents of two definitions are indistinguished.When $S^{(n)} = \{K_i : k \le i \le k\}$(the number of K_k is m,and km=n) ,K_k-factor is a special case of $S^{(n)} = \{K_i : 1 \le i \le n\}$-factor.In [10], E J. Farrell and J M.Guo discussed the characterizing of matching polynomials (or 1-factors). In [13], M.N.Ellingham, YunsunNam Heinz-Jürgen Voss presented connected (g,f)-factors. In [14], Potra Johann derived the structure of graphs with a unique k-factor. In [15], Atsushi Kaneko gained a necessary and sufficient condition for the existence of a path factor. The lectures of 1-factors or (g,f)-factors have been belonged to the field of existences, here for us, we consider principally enumeration of $S^{(n)} = \{K_i : 1 \le i \le n\}$-factors, our methods are from combinatorial idea such as recurrence relations or inverse problems. We research the significance of $S^{(n)} = \{K_i : 1 \le i \le n\}$-factors as follows: 1. enumeration of graph theory; 2. combinatorial values(main combinatorial identities); 3. computer and code (regular-m-furcating tree, special regular 2-furcating tree); 4. other aspects. (the mean colour numbers $\mu(G)$,we complete these papers,see LiMin Yang and TianMing Wang [16]and[17].)

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Definition 1.1. Let $S^{(n)} = \{K_i : 1 \le i \le n\}$, $n \ge 1$, and K_i is a complete graph with i vertices, if M is a subgraph of a graph G, and any component of M is isomorphic to some element of $S^{(n)} = \{K_i : 1 \le i \le n\}$, then M is called one $S^{(n)}$-subgraph, if M is a spanning subgraph of the graph G, then M is called one $S^{(n)} = \{K_i : 1 \le i \le n\}$-factor .

Definition 1.2. If M is a spanning subgraph of the graph G,each component of M is all isomorphic to K_k, then M is called one K_k-factor.

Complexity: It is easy to see that $I(G, x)$ is NP-hard to compute.(see [1])

Let$N(G, k)$denote the number of $S^{(n)} = \{K_i : 1 \le i \le n\}$-factors with exactly k components, $A(G)$ is the number of all $S^{(n)} = \{K_i : 1 \le i \le n\}$-factors, namely, $A(G) = \sum_{k=1}^{n} N(G, k)$. In [1], by partition-map-relation, the authors obtained the counting formula of $N(K_n, k)$. In [2], in the the use of combinatorial convolutions, the authors derived the counting formula of $A(K_n)$. In [3], the author has found myself the recurrence relation by covering methods, and gained the formula of $A(0 \odot C_n)$. In [4], we proved the recurrence formula of $S^{(n)}$-factors of regular m-furcating tree by analyzing the relation among m, i and t. They have been very difficult problems and heat contents that we count $N(G, k)$ and $A(G)$. In this paper, by combinatorial methods, the authors solve the representing formula between $N(G, k)$ and coefficients of the chromatic polynomial of $\bar{G}$, and the representing formula between $A(G)$ and coefficients of the chromatic polynomial of $\bar{G}$, present the counting formulas of classes of graphs $N(G, k)$ and $A(G)$, and some combinatorial identities on $N(G, k)$ are given.Finally, we solve the very difficult and heat problem on the the counting formula of K_d-factors in any complete d-partite $G = K_{n,n,\cdots,n}$,the counting formula of the number of covers on shortest circles, as well as the explicit formula $\mu(K_n^d)$of the mean numbers of K_n^d.

2. Basic Lemmas

Here we will denote that $\alpha(G, k)$ is the number of partitions of $V(G)$ into exactly k non-empty independent sets of any graph G.

Lemma 2.1 ([1]). *Let K_n be a complete graph with n vertices. Then the number of $S^{(n)}$-factors with exactly k components in K_n :*

$$N(K_n, k) = \sum_{\sum_{i=1}^{n} ib_i = n,\ \sum_{i=1}^{n} b_i = k} \frac{n!}{b_1!} \prod_{i \ge 2} \frac{1}{b_i!(i!)^{b_i}} .$$

Lemma 2.2 ([1]). *Suppose K_n is a complete graph with n vertices, then the number of $S^{(n)}$-factors in K_n :*

$$A(K_n) = \frac{1}{e} \sum_{k=1}^{\infty} \frac{k^n}{k!} .$$

Lemma 2.3 ([5])**.** *If $s(n,k)$ is the Stirling number of the first kind, $S(n,k)$ is the Stirling number of the second kind, then $[s(n,k)] = [S(n,k)]^{-1}$, where*

$$[s(n,k)] = \begin{pmatrix} s(1,1) & s(1,2) & \cdots & s(1,n) \\ s(2,1) & s(2,2) & \cdots & s(2,n) \\ \cdots & \cdots & \cdots & \cdots \\ s(n-1,1) & s(n-1,2) & \cdots & s(n-1,n) \\ s(n,1) & s(n,2) & \cdots & s(n,n) \end{pmatrix}_{n\times n},$$

$$[S(n,k)] = \begin{pmatrix} S(1,1) & S(1,2) & \cdots & S(1,n) \\ S(2,1) & S(2,2) & \cdots & S(2,n) \\ \cdots & \cdots & \cdots & \cdots \\ S(n-1,1) & S(n-1,2) & \cdots & S(n-1,n) \\ S(n,1) & S(n,2) & \cdots & S(n,n) \end{pmatrix}_{n\times n}.$$

Lemma 2.4 ([5])**.** *If $S(n,k)$ is the Stirling number of the second kind, then*

$$N(K_n, k) = S(n,k)\ .$$

Lemma 2.5 ([7])**.** *Suppose $\bar{G}$ is a complementary graph of G, then the chromatic polynomial of G is as the following*

$$f(G,t) = \sum_{r=1}^{n} [\sum_{k=1}^{n} N(\bar{G},k)s(k,r)]\, t^r,$$

where $N(\bar{G},k)$ denotes the number of $S^{(n)}$-factors with exactly k components in $\bar{G}$.

Lemma 2.6. *Suppose G is not an empty graph, then*

$$\sum_{r=1}^{n}\sum_{k=1}^{n} N(\bar{G},k)s(k,r) = 0\ .$$

Proof. The result follows from $f(G,1) = \sum_{r=1}^{n}\sum_{k=1}^{n} N(\bar{G},k)s(k,r)$ and $f(G,1) = 0$. ☐

3. The representing formula of $N(G,k)$

In this section,our main result is the representing formula of $N(G,k)$ as follows.

Theorem 3.1. *If G is a graph with n vertices, and the chromatic polynomial of of the complementary graph $\bar{G}$ of G is $f(\bar{G},t) = \sum_{p=1}^{n} Y_p t^p$, then the number of $S^{(n)}$-factors with exactly k components is the following*

$$N(G,k) = \sum_{p=k}^{n} N(K_p,k)Y_p\ ,$$

where

$$N(K_p\,,k) = \sum_{\sum_{i=1}^{p} ib_i = p,\ \sum_{i=1}^{p} b_i = k} \frac{p!}{b_1!}\prod_{i\geq 2}\frac{1}{b_i!(i!)^{b_i}},\ 1\leq p\leq n,\ 1\leq k\leq n\ .$$

Proof. By combinatorial and algebraic methods, we will prove the theorem. By lemma 2.5, we have the chromatic polynomial of $\bar{G}$

$$f(\bar{G},t)=\sum_{r=1}^{n}[\sum_{k=1}^{n}N(G,k)s(k,r)]\,t^r$$

and

$$f(\bar{G},t)=\sum_{p=1}^{n}Y_p t^p,$$

so that we have the equal systems

$$\begin{cases}\sum\limits_{k=1}^{n}N(G,k)s(k,1)=Y_1\,,\\ \sum\limits_{k=1}^{n}N(G,k)s(k,2)=Y_2\,,\\ \qquad\cdots\quad\cdots\\ \sum\limits_{k=1}^{n}N(G,k)s(k,n)=Y_n,\end{cases} \tag{3.1}$$

$$\begin{cases}s(1,1)N(G,1)+s(2,1)N(G,2)+\cdots+s(n,1)N(G,n)=Y_1\,,\\ s(1,2)N(G,1)+s(2,2)N(G,2)+\cdots+s(n,2)N(G,n)=Y_2\,,\\ \qquad\cdots\quad\cdots\quad\cdots\\ s(1,n-1)N(G,1)+s(2,n-1)N(G,2)+\cdots+s(n,n-1)N(G,n)=Y_{n-1}\,,\\ s(1,n)N(G,1)+s(2,n)N(G,2)+\cdots+s(n,n)N(G,n)=Y_n\,,\end{cases} \tag{3.2}$$

By Lemma 2.3, we have $[s(n,k)]=[S(n,k)]^{-1}$ (Inverse relations derived by combinatorial idea,see[5]).

$$[s(n,k)][S(n,k)]=E,\qquad [S(n,k)]'[s(n,k)]'=E$$

$$\begin{pmatrix}S(1,1)&S(2,1)&\cdots&S(n-1,1)&S(n,1)\\ S(1,2)&S(2,2)&\cdots&S(n-1,2)&S(n,2)\\ \cdots&\cdots&\cdots&\cdots&\cdots\\ S(1,n)&S(2,n)&\cdots&S(n-1,n)&S(n,n)\end{pmatrix}\begin{pmatrix}s(1,1)&s(2,1)&\cdots&s(n-1,1)&s(n,1)\\ s(1,2)&s(2,2)&\cdots&s(n-1,2)&s(n,2)\\ \cdots&\cdots&\cdots&\cdots&\cdots\\ s(1,n)&s(2,n)&\cdots&s(n-1,n)&s(n,n)\end{pmatrix}=E$$

$$\begin{pmatrix}s(1,1)&s(2,1)&\cdots&s(n-1,1)&s(n,1)\\ s(1,2)&s(2,2)&\cdots&s(n-1,2)&s(n,2)\\ \cdots&\cdots&\cdots&\cdots&\cdots\\ s(1,n)&s(2,n)&\cdots&s(n-1,n)&s(n,n)\end{pmatrix}^{-1}=\begin{pmatrix}S(1,1)&S(2,1)&\cdots&S(n-1,1)&S(n,1)\\ S(1,2)&S(2,2)&\cdots&S(n-1,2)&S(n,2)\\ \cdots&\cdots&\cdots&\cdots&\cdots\\ S(1,n)&S(2,n)&\cdots&S(n-1,n)&S(n,n)\end{pmatrix} \tag{3.3}$$

By (3.2), we have the equality

$$\begin{pmatrix}N(G,1)\\ N(G,2)\\ \cdots\\ N(G,n-1)\\ N(G,n)\end{pmatrix}=\begin{pmatrix}s(1,1)&s(2,1)&\cdots&s(n,1)\\ s(1,2)&s(2,2)&\cdots&s(n,2)\\ \cdots&\cdots&\cdots&\cdots\\ s(1,n-1)&s(2,n-1)&\cdots&s(n,n-1)\\ s(1,n)&s(2,n)&\cdots&s(n,n)\end{pmatrix}^{-1}\begin{pmatrix}Y_1\\ Y_2\\ \cdots\\ Y_{n-1}\\ Y_n\end{pmatrix}$$

By (3.3), there is the equal n-tuples

$$\begin{pmatrix} N(G,1) \\ N(G,2) \\ \cdots \\ N(G,n-1) \\ N(G,n) \end{pmatrix} = \begin{pmatrix} S(1,1) & S(2,1) & \cdots & S(n,1) \\ S(1,2) & S(2,2) & \cdots & S(n,2) \\ \cdots & \cdots & \cdots & \cdots \\ S(1,n-1) & S(2,n-1) & \cdots & S(n,n-1) \\ S(1,n) & S(2,n) & \cdots & S(n,n) \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ \cdots \\ Y_{n-1} \\ Y_n \end{pmatrix}$$

By Lemma 2.4,there is the formula $S(n,k) = N(K_n,k)$. Then

$$\begin{pmatrix} N(G,1) \\ N(G,2) \\ \cdots \\ N(G,n-1) \\ N(G,n) \end{pmatrix} = \begin{pmatrix} N(K_1,1) & N(K_2,1) & \cdots & N(K_{n-1},1) & N(K_n,1) \\ N(K_1,2) & N(K_2,2) & \cdots & N(K_{n-1},2) & N(K_n,2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ N(K_1,n-1) & N(K_2,n-1) & \cdots & N(K_{n-1},n-1) & N(K_n,n-1) \\ N(K_1,n) & N(K_2,n) & \cdots & N(K_{n-1},n) & N(K_n,n) \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ \cdots \\ Y_{n-1} \\ Y_n \end{pmatrix}$$

When $k > n$, $N(K_n,k) = 0$, then there is the equality

$$\begin{pmatrix} N(G,1) \\ N(G,2) \\ \cdots \\ N(G,n-1) \\ N(G,n) \end{pmatrix} = \begin{pmatrix} N(K_1,1) & N(K_2,1) & \cdots & N(K_{n-1},1) & N(K_n,1) \\ 0 & N(K_2,2) & \cdots & N(K_{n-1},2) & N(K_n,2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & N(K_{n-1},n-1) & N(K_n,n-1) \\ 0 & 0 & \cdots & 0 & N(K_n,n) \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ \cdots \\ Y_{n-1} \\ Y_n \end{pmatrix}, \tag{3.4}$$

so the representing formula of $N(G,k)$ is derived from (3.4)

$N(G,k) = N(K_k,k)Y_k + N(K_{k+1},k)Y_{k+1} + \cdots + N(K_{n-1},k)Y_{n-1} + N(K_n,k)Y_n$,

for $1 \le k \le n$.*Namely*, $N(G,k) = \sum_{p=k}^{n} N(K_p,k)Y_p$, where $Y_p (1 \le p \le n)$ are coefficients of the chromatic polynomial $f(\bar{G},t)$ and

$$N(K_p\,,k) = \sum_{\sum_{i=1}^{p} ib_i = p,\ \sum_{i=1}^{p} b_i = k} \frac{p!}{b_1!} \prod_{i \ge 2} \frac{1}{b_i!(i!)^{b_i}}$$

(obtained by Lemma 2.1) here $1 \le p \le n$, $1 \le k \le n$. □

4. The representing formula of $A(G)$

In the segment, we characterize the representing formula of $A(G)$.

Theorem 4.1. *If G is the graph with n vertices,and the chromatic polynomial of the complementary graph $\bar{G}$ of G is $f(\bar{G},t) = \sum_{p=1}^{n} Y_p t^p$,then the number of all $S^{(n)}$-factors in G is the following*

$$A(G) = \frac{1}{e} \sum_{p=1}^{n} \sum_{k=1}^{\infty} \frac{k^p}{k!} Y_p \ .$$

Proof. By Theorem 3.1,there are equalities:

$$\begin{cases} N(G,1) = N(K_1,1)Y_1 + N(K_2,1)Y_2 + \cdots + N(K_{n-1},1)Y_{n-1} + N(K_n,1)Y_n \ , \\ N(G,2) = N(K_2,2)Y_2 + \cdots + N(K_{n-1},2)Y_{n-1} + N(K_n,2)Y_n \ , \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ N(G,n-1) = N(K_{n-1},n-1)Y_{n-1} + N(K_n,n-1)Y_n \ , \\ N(G,n) = N(K_n,n)Y_n \ , \end{cases}$$

$$\begin{aligned} A(G) &= \sum_{k=1}^{n} N(G,k) \\ &= N(K_1,1)Y_1 + [N(K_2,1) + N(K_2,2)]Y_2 + \cdots \\ &\quad + [N(K_{n-1},1) + N(K_{n-1},2) + \cdots + N(K_{n-1},n-1)]Y_{n-1} \\ &\quad + [N(K_n,1) + N(K_n,2) + \cdots + N(K_n,n-1) + N(K_n,n)]Y_n \\ &= A(K_1)Y_1 + A(K_2)Y_2 + \cdots + A(K_{n-1})Y_{n-1} + A(K_n)Y_n \\ &= \sum_{p=1}^{n} A(K_p)Y_p \ , \end{aligned}$$

where $A(K_p) = \sum\limits_{k=1}^{p} N(K_p\,,k)$, for $1 \leq p \leq n$.

By Lemma 2.2,

$$A(K_p) = \frac{1}{e}\sum_{k=1}^{\infty}\frac{k^p}{k!} \ , \ 1 \leq p \leq n \ .$$

Then

$$A(G) = \sum_{p=1}^{n}\frac{1}{e}\sum_{k=1}^{\infty}\frac{k^p}{k!}Y_p = \frac{1}{e}\sum_{p=1}^{n}\sum_{k=1}^{\infty}\frac{k^p}{k!}Y_p \ ,$$

where $Y_p(\ 1 \leq p \leq n)$ are coefficients of the chromatic polynomial $f(\bar{G},t)$. □

5. Nnumeration of classes of graphs

In the section, we present classes of graphs $N(G,k)$ and $A(G)$. Specially, enumeration of complementary trees is solved.

Theorem 5.1. *Suppose G is a $(n-2)$-regular graph with n (even $2m$) vertices, then the number of $S^{(n)}$-factors with exactly k components is the following*

$$N(G,k) = \sum_{p=k}^{n} N(K_p,k)Y_p \ ,$$

where for $1 \leq p \leq m-1, \ Y_p = 0$*, for* $m \leq p \leq 2m, \ Y_p = (-1)^{2m-p}\binom{m}{2m-p}$,

$$N(K_p\,,k) = \sum_{\sum\limits_{i=1}^{p} ib_i = p, \ \sum\limits_{i=1}^{p} b_i = k} \frac{p!}{b_1!}\prod_{i\geq 2}\frac{1}{b_i!(i!)^{b_i}} \ , \ 1 \leq p \ , \ k \leq n \ .$$

Proof. Because G is a $(n-2)$-regular graph, $\bar{G}$ is 1-regular graph, including the number m of K_2, then $\bar{G} = K_2 \bigcup K_2 \bigcup \cdots \bigcup K_2$.

The chromatic polynomial of the complementary graph $\bar{G}$ is as follows

$$\begin{aligned} f(\bar{G}, t) &= f(K_2, t) f(K_2, t) \cdots f(K_2, t) = f^m(K_2, t) = t^m (t-1)^m \\ &= t^m \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} t^i = \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} t^{m+i} \\ &= \sum_{p=m}^{2m} (-1)^{2m-p} \binom{m}{2m-p} t^p \ , \end{aligned}$$

$Y_p = 0,\ 1 \le p \le m-1;\ Y_p = (-1)^{2m-p} \binom{m}{2m-p},\ m \le p \le 2m.$

By Theorem 3.1,we have the equality

$$N(G, k) = \sum_{p=k}^{n} N(K_p, k) Y_p \ ,$$

where for $1 \le p \le m-1,\ Y_p = 0$, for $m \le p \le 2m,\ Y_p = (-1)^{2m-p} \binom{m}{2m-p}$,

$$N(K_p\,, k) = \sum_{\sum_{i=1}^{p} i b_i = p,\ \sum_{i=1}^{p} b_i = k} \frac{p!}{b_1!} \prod_{i \ge 2} \frac{1}{b_i!(i!)^{b_i}} \ ,\ 1 \le p\ ,\ k \le n\ .$$

□

Theorem 5.2. *Suppose G is a $(n-2)$-regular graph with n (even $2m$) vertices, then*

$$A(G) = \frac{1}{e} \sum_{p=m}^{2m} \sum_{k=1}^{\infty} (-1)^{2m-p} \binom{m}{2m-p} \frac{k^p}{k!} \ .$$

Proof. By Theorem 4.1 $A(G) = \sum_{p=1}^{n} A(K_p) Y_p$, and by Theorem 5.1, when $1 \le p \le m-1,\ Y_p = 0$, when $m \le p \le n = 2m,\ Y_p = (-1)^{2m-p} \binom{m}{2m-p}$, finally, there is the equality

$$A(G) = \sum_{p=m}^{2m} (-1)^{2m-p} \binom{m}{2m-p} A(K_p) \ ,$$

and by Lemma 2.2 $A(K_p) = \frac{1}{e} \sum_{k=1}^{\infty} \frac{k^p}{k!}$, for $m \le p \le n$. So we have the formula

$$A(G) = \frac{1}{e} \sum_{p=m}^{2m} \sum_{k=1}^{\infty} (-1)^{2m-p} \binom{m}{2m-p} \frac{k^p}{k!} \ .$$

□

Theorem 5.3. *Suppose G is a $(n-3)$-regular graph with n vertices, $n \geq 6$ and $\bar{G} \cong C_n$, then*

$$N(G,k) = \sum_{p=k}^{n} (-1)^{n-p} \binom{n}{p} N(K_p\,,k)\ ,$$

where

$$N(K_p\,,k) = \sum_{\sum_{i=1}^{p} ib_i = p,\ \sum_{i=1}^{p} b_i = k} \frac{p!}{b_1!} \prod_{i \geq 2} \frac{1}{b_i!(i!)^{b_i}}\ ,\ 2 \leq p\ ,\ k \leq n\ .$$

Proof. If G is a $(n–3)$-regular graph, then $\bar{G}$ is a 2-regular graph with n vertices,and $\bar{G} \cong C_n$.

$$\begin{aligned} f(\bar{G},t) &= f(C_n,t) = (t-1)^n + (-1)^n(t-1) \\ &= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} t^n + (-1)^n(t-1) \\ &= [(-1)^{n-1}\binom{n}{1} + (-1)^n]t + (-1)^{n-2}\binom{n}{2}t^2 \\ &\quad + \cdots + (-1)^{n-p}\binom{n}{p}t^p \quad + \cdots + (-1)^1\binom{n}{n-1}t^{n-1} + t^n\ , \end{aligned}$$

$$Y_1 = (-1)^{n-1}\binom{n-1}{1},\ Y_2 = (-1)^{n-2}\binom{n}{2}, Y_3 = (-1)^{n-3}\binom{n}{3},$$

$$\cdots,\ Y_p = (-1)^{n-p}\binom{n}{p}, \cdots,\ Y_{n-1} = -\binom{n}{n-1},\ Y_n = 1.$$

By Theorem 3.1, then $N(G,k) = \sum_{p=k}^{n} N(K_p\,,k)Y_p$. For $k = 1$, G is a $(n-3)$-regular graph, $G \neq K_n$, then $N(G,1) = 0$; For $2 \leq k \leq n$,

$$N(G,k) = \sum_{p=k}^{n} (-1)^{n-p} \binom{n}{p} N(K_p\,,k)\ ,$$

where

$$N(K_p\,,k) = \sum_{\sum_{i=1}^{p} ib_i = p,\ \sum_{i=1}^{p} b_i = k} \frac{p!}{b_1!} \prod_{i \geq 2} \frac{1}{b_i!(i!)^{b_i}}\ ,\ 2 \leq p\ ,\ k \leq n\ .$$

Note Suppose G is a $(n–3)$-regular graph, $\bar{G} = C_{i_1} \bigcup C_{i_2} \bigcup \cdots \bigcup C_{i_m}$, $i_1 + i_2 + \cdots + i_m = n$, analogous to Theorem 5.3, we would have the corresponding result. □

Theorem 5.4. *Suppose G is a (n-3)-regular graph with n vertices, $n \geq 6$ and $\bar{G} \cong C_n$, then*

$$A(G) = (-1)^{n-1}\binom{n-1}{1} + \frac{1}{e}\sum_{p=2}^{n}\sum_{k=1}^{\infty}(-1)^{n-p}\binom{n}{p}\frac{k^p}{k!}\ .$$

Proof. Because G is a $(n-3)$-regular graph,$\bar{G} \cong C_n$ and by Theorem 5.3,then

$$\begin{aligned} f(\bar{G},t) &= f(C_n,t) \\ &= (-1)^{n-1}\binom{n-1}{1}t + (-1)^{n-2}\binom{n}{2}t^2 + \cdots + (-1)^{n-p}\binom{n}{p}t^p \\ &\quad + \cdots + (-1)\binom{n}{1}t^{n-1} + t^n \ , \end{aligned}$$

$$Y_1 = (-1)^{n-1}\binom{n-1}{1},\ Y_2 = (-1)^{n-2}\binom{n}{2},\ \cdots,$$

$$Y_p = (-1)^{n-p}\binom{n}{p}, \cdots,\ Y_{n-1} = (-1)\binom{n}{1},\ Y_n = 1.$$

By Theorem 4.1, we have

$$A(G) = \sum_{p=1}^{n} A(K_p)Y_p = (-1)^{n-1}\binom{n-1}{1} + \sum_{p=2}^{n}(-1)^{n-p}\binom{n}{p}A(K_p) \ ,$$

and by Lemma 2.2

$$A(K_p) = \frac{1}{e}\sum_{k=1}^{\infty}\frac{k^p}{k!},\ 2 \le p \le n \ ,$$

finally

$$A(G) = (-1)^{n-1}\binom{n-1}{1} + \frac{1}{e}\sum_{p=2}^{n}\sum_{k=1}^{\infty}(-1)^{n-p}\binom{n}{p}\frac{k^p}{k!} \ .$$

Note Let G be a (n–3)-regular graph, $\bar{G} = C_{i_1}\bigcup C_{i_2}\bigcup\cdots\bigcup C_{i_m},\ i_1+i_2+\cdots+i_m = n$, we would also have the corresponding result. □

Theorem 5.5. *If T is a tree with n vertices, and $\bar{T}$ is a complementary complementary graph of T, then*

$$N(\bar{T},k) = \sum_{p=k}^{n}(-1)^{n-p}\binom{n-1}{p-1}N(K_p\,,k) \ ,$$

where

$$N(K_p\,,k) = \sum_{\sum_{i=1}^{p} ib_i=p,\ \sum_{i=1}^{p} b_i=k}\frac{p!}{b_1!}\prod_{i\ge 2}\frac{1}{b_i!(i!)^{b_i}}\ ,\ 2 \le p\ ,\ k \le n \ .$$

Proof. Let G be a tree with n vertices. Then

$$\begin{aligned} f(T,t) &= t(t-1)^{n-1} = t\sum_{k=0}^{n-1}(-1)^{n-1-k}\binom{n-1}{k}t^k \\ &= \sum_{k=0}^{n-1}(-1)^{n-1-k}\binom{n-1}{k}t^{k+1} \\ &= (-1)^{n-1}t + (-1)^{n-2}\binom{n-1}{1}t^2 + (-1)^{n-3}\binom{n-1}{2}t^3 + \cdots \\ &\quad +(-1)^{n-p}\binom{n-1}{p-1}t^p \quad + \cdots + (-1)\binom{n-1}{n-2}t^{n-1} + t^n \ . \end{aligned}$$

When $1 \leq p \leq n$, $Y_p = (-1)^{n-p}\binom{n-1}{p-1}$. By Theorem 3.1 $N(\bar{T},k) = \sum_{p=k}^{n} N(K_p\,,k)Y_p$. Then

$$N(\bar{T},k) = \sum_{p=k}^{n}(-1)^{n-p}\binom{n-1}{p-1}N(K_p\,,k) \ ,$$

where

$$N(K_p\,,k) = \sum_{\sum_{i=1}^{p} ib_i=p,\ \sum_{i=1}^{p} b_i=k} \frac{p!}{b_1!}\prod_{i\geq 2}\frac{1}{b_i!(i!)^{b_i}} \ , \ 2 \leq p \ , \ k \leq n \ .$$

□

Theorem 5.6. *If T is a tree with n vertices, and $\bar{T}$ is a complementary graph of T, then*

$$A(\bar{T}) = \frac{1}{e}\sum_{p=1}^{n}\sum_{k=1}^{\infty}(-1)^{n-p}\binom{n-1}{p-1}\frac{k^p}{k!} \ .$$

Proof. By Theorem 5.5,

$$f(T,t) = (-1)^{n-1}t + (-1)^{n-2}\binom{n-1}{1}t^2 + \cdots + (-1)^{n-p}\binom{n-1}{p-1}t^p + \cdots + (-1)\binom{n-1}{n-2}t^{n-1} + t^n \ ,$$

then

$$Y_p = (-1)^{n-p}\binom{n-1}{p-1}, \ 1 \leq p \leq n \ .$$

By Theorem 4.1,

$$A(\bar{T}) = \sum_{p=1}^{n} A(K_p)Y_p = \sum_{p=1}^{n}(-1)^{n-p}\binom{n-1}{p-1}A(K_p) \ ,$$

and

$$A(K_p) = \frac{1}{e}\sum_{k=1}^{\infty}\frac{k^p}{k!},\ 1 \le p \le n\ ,$$

so we derive the formula

$$A(\bar{T}) = \frac{1}{e}\sum_{p=1}^{n}\sum_{k=1}^{\infty}(-1)^{n-p}\binom{n-1}{p-1}\frac{k^p}{k!}\ .$$

Here we solve the explicit formula of any complementary tree $\bar{T}$ on enumeration of $S^{(n)}$-factors. □

Theorem 5.7. *If G is a windgraph K_n^d, K_1 is a meet vertex of K_n with the number d, (or K_1 is a intersection vertex of the number d K_n), then*

$$N(K_n^d, k) = \sum_{p=k}^{d(n-1)+1} N(K_p, k)Y_p\ ,$$

where

$$Y_1 = 0,\ Y_p = \sum_{k=1}^{d(n-1)}[\sum_{l_1+\cdots+l_d=k} S(n,l_1)\cdots S(n,l_d)]s(k,p-1)$$

for $2 \le p \le d(n-1)+1$.

Proof. Let G be K_n^d. Then $\bar{G} = K_1 \bigcup H$, H is a complete complete d-partite graph, thus assume that $H = K_{n-1,n-1,\cdots,n-1}$ with the number d of $n-1$, say.

$$\begin{aligned} N(\bar{H}, k) &= \sum_{l_1+\cdots+l_d=k} N(K_{n-1}, l_1)N(K_{n-1}, l_2)\cdots N(K_{n-1}, l_d) \\ &= \sum_{l_1+\cdots+l_d=k} S(n-1, l_1)S(n-1, l_2)\cdots S(n-1, l_d)\ . \end{aligned}$$

The chromatic polynomial of $\bar{G}$ is as follows

$$f(\bar{G}, t) = f(K_1 \cup H, t) = f(K_1, t)f(H, t) = tf(H, t)\ .$$

By Lemma 2.5, then we obtain the chromatic polynomial $f(\bar{G}, t)$,

$$\begin{aligned} f(\bar{G}, t) &= t\sum_{r=1}^{d(n-1)}[\sum_{k=1}^{d(n-1)} N(\bar{H}, k)s(k, r)]t^r \\ &= t\sum_{r=1}^{d(n-1)}\sum_{k=1}^{d(n-1)}[\sum_{l_1+\cdots+l_d=k}\prod_{i=1}^{d} S(n-1, l_i)]s(k, r)t^r \\ &= \sum_{r=1}^{d(n-1)}\sum_{k=1}^{d(n-1)}[\sum_{l_1+\cdots+l_d=k}\prod_{i=1}^{d} S(n-1, l_i)]s(k, r)t^{r+1}\ . \end{aligned}$$

Coefficients of $f(\bar{G}, t)$ are the following

$$Y_1 = 0,\ Y_p = \sum_{k=1}^{d(n-1)}[\sum_{l_1+\cdots+l_d=k}\prod_{i=1}^{d} S(n-1, l_i)]s(k, p-1),\ 2 \le p \le d(n-1)+1\ .$$

By Theorem 3.1, we have the result

$$N(K_n^d, k) = \sum_{p=k}^{d(n-1)+1} N(K_p, k) Y_p \ ,$$

where

$$Y_1 = 0,\ Y_p = \sum_{k=1}^{d(n-1)} [\sum_{l_1+\cdots+l_d=k} \prod_{i=1}^{d} S(n-1, l_i)] s(k, p-1),\ 2 \le p \le d(n-1)+1 \ ,$$

$d(n-1)+1$ is the number of vertices of K_n^d . □

Theorem 5.8. *Suppose G is a windgraph K_n^2, special $d = 2$, then*

$$N(K_n^2, k) = \sum_{p=k}^{2n-1} N(K_p\,, k) \sum_{k=1}^{2n-2} \sum_{l_1+l_2=k} S(n-1, l_1) S(n-1, l_2) s(k, p-1),$$

and $2 \le k \le 2n-1$.

Proof. Omitted. □

Theorem 5.9. *If G is a windgraph K_n^d, K_1 is a meet vertex of K_n with the number d, then*

$$A(K_n^d) = \frac{1}{e} \sum_{p=2}^{d(n-1)+1} \sum_{k=1}^{\infty} \frac{k^p}{k!} \sum_{k=1}^{d(n-1)} [\sum_{l_1+\cdots+l_d=k} \prod_{i=1}^{d} S(n-1, l_i)] s(k, p-1) \ ,$$

where $d(n-1)+1$ is the number of vertices of K_n^d.

Proof. Because G is a windgraph K_n^d,and the number of vertices is $d(n-1)+1$,by Theorem 5.7,then the chromatic polynomial of $\bar{G}$ is

$$f(\bar{G}, t) = \sum_{r=1}^{d(n-1)} \{ \sum_{k=1}^{d(n-1)} [\sum_{l_1+\cdots+l_d=k} \prod_{i=1}^{d} S(n-1, l_i)] s(k, r) \} t^{r+1} \ .$$

We have coefficients of $f(\bar{G}, t)$ are

$$Y_1 = 0,\ Y_p = \sum_{k=1}^{d(n-1)} [\sum_{l_1+\cdots+l_d=k} \prod_{i=1}^{d} S(n-1, l_i)] s(k, p-1),\ 2 \le p \le d(n-1)+1 \ ,$$

By Theorem 4.1,

$$A(K_n^d) = \sum_{p=1}^{d(n-1)+1} A(K_p) Y_p$$

and by Lemma 2.2

$$A(K_p) = \frac{1}{e} \sum_{k=1}^{\infty} \frac{k^p}{k!},\ 2 \le p \le n \ ,$$

namely,

$$A(K_n^d) = \frac{1}{e} \sum_{p=2}^{d(n-1)+1} \sum_{k=1}^{\infty} \frac{k^p}{k!} Y_p \ ,$$

then we have the result

$$A(K_n^d) = \frac{1}{e}\sum_{p=2}^{d(n-1)+1}\sum_{k=1}^{\infty}\frac{k^p}{k!}\sum_{k=1}^{d(n-1)}[\sum_{l_1+\cdots+l_d=k}\prod_{i=1}^{d} S(n-1,l_i)]s(k,p-1)\ ,$$

where $d(n-1)+1$ is the number of vertices of K_n^d. So far, we have gained the

explicit formulas of classes of graphs on enumeration of $S^{(n)}$-factors. □

6. Combinatorial identities for $N(G,k)$

Theorem 6.1. *If $N(K_p,k)$ is the number of $S^{(n)}$-factors with exactly k components in K_p , and $n \in N$, $1 \le p\,,k \le n$, then*

$$\sum_{r=1}^{n}\sum_{k=1}^{n}\sum_{p=k}^{n}(-1)^{n-p}\binom{n-1}{p-1}N(K_p,k)\times$$
$$\sum_{0\le h\le k-r}(-1)^h\binom{k-1+h}{k-r+h}\binom{2k-r}{k-r-h}N(K_{k-r+h},h)=0\ .$$

Proof. Let $f(G,t)$ be the chromatic polynomial of G. Then

$$f(G,t) = \sum_{r=1}^{n}[\sum_{k=1}^{n}N(\bar{G},k)\times \tag{6.1}$$
$$\sum_{0\le h\le k-r}(-1)^h\binom{k-1+h}{k-r+h}\binom{2k-r}{k-r-h}N(K_{k-r+h},h)]t^r \quad \text{(see [7])}\ ,$$

where K_{k-r+h} is a complete graph with $k-r+h$ vertices. When $t=1$, then

$$f(G,1)=\sum_{r=1}^{n}\sum_{k=1}^{n}N(\bar{G},k)\sum_{0\le h\le k-r}(-1)^h\binom{k-1+h}{k-r+h}\binom{2k-r}{k-r-h}N(K_{k-r+h},h)\ .$$

On the other hand, if G is not an empty graph,let $t=1$,then $f(G,1)$=0. So G is not an empty graph

$$\sum_{r=1}^{n}\sum_{k=1}^{n}N(\bar{G},k)\sum_{0\le h\le k-r}(-1)^h\binom{k-1+h}{k-r+h}\binom{2k-r}{k-r-h}N(K_{k-r+h},h)=0\ .$$

Suppose G is a tree T with n vertices, and by Theorem 5.5, then we have

$$N(\bar{G},k)=\sum_{p=k}^{n}(-1)^{n-p}\binom{n-1}{p-1}N(K_p\,,k)\ .$$

Finally, there exists a combinatorial identity for $N(K_p\,,k)$:

$$\sum_{r=1}^{n}\sum_{k=1}^{n}\sum_{p=k}^{n}(-1)^{n-p}\binom{n-1}{p-1}N(K_p,k)\times \tag{6.2}$$
$$\sum_{0\le h\le k-r}(-1)^h\binom{k-1+h}{k-r+h}\binom{2k-r}{k-r-h}N(K_{k-r+h},h)=0\ .$$

where

$$N(K_p,k)=\sum_{\sum_{i=1}^{p} ib_i=p,\ \sum_{i=1}^{p} b_i=k}\frac{p!}{b_1!}\prod_{i\geq 2}\frac{1}{b_i!(i!)^{b_i}}\ ,\ 1\leq p\ ,\ k\leq n\ .$$

□

Theorem 6.2. *If $S(n,k)$ is the Stirling number of the second kind, $s(k,r)$ is the Stirling number of the first kind, then there exists a combinatorial identity*

$$\sum_{r=1}^{dn}\sum_{k=1}^{dn}\sum_{l_1+\cdots+l_d=k}\prod_{i=1}^{d}S(n,l_i)s(k,r)=0\ .$$

Proof. Let G be a complete d-partite graph, $G=K_{n,n,\cdots,n}$ with with the number d of n, dn is the number of vertices of G,

$$N(\bar{G},k)=\sum_{l_1+\cdots+l_d=k}N(K_n,l_1)N(K_n,l_2)\cdots N(K_n,l_d)\ .$$

By Lemma 2.4,

$$N(\bar{G},k)=\sum_{l_1+\cdots+l_d=k}S(n,l_1)S(n,l_2)\cdots S(n,l_d)\ .$$

By Lemma 2.6,then there exists combinatorial identity as follows:

$$\sum_{r=1}^{dn}\sum_{k=1}^{dn}\sum_{l_1+\cdots+l_d=k}\prod_{i=1}^{d}S(n,l_i)s(k,r)=0\ .$$

□

Corollary 6.3. *There exists a combinatorial identity*

$$\sum_{r=1}^{2n}\sum_{k=1}^{2n}\sum_{l_1+l_2=k}S(n,l_1)S(n,l_2)s(k,r)=0\ ,$$

where $S(n,k)$ is the Stirling number of the second kind, $s(k,r)$ is the Stirling number of the first kind.

Proof. Let G be a complete 2-partite graph $G=K_{n,n}$.Then

$$N(\bar{G},k)=\sum_{l_1+l_2=k}S(n,l_1)S(n,l_2)\ .$$

By Theorem 6.2, the proof is proved. □

7. The Counting formula of K_d-factors in any complete d-partite graph

In the section, we give the explicit formula of enumeration of K_d-factors in any complete d-partite graph. It is a very difficult and value problem that so far these problems of factors have been researched, involving 1-factors ,(g,f)-factors and $S^{(n)}$-factors.

Theorem 7.1. *Suppose G is a complete d-partite graph $G = K_{n,n,\cdots,n}$ with dn vertices,then the explicit formula of the number of all K_d-factors in G is as follows:*

$$N(G,n) = \sum_{p=n}^{dn} N(K_p,n) \sum_{\sum_{1\le j\le d} l_j = p} \prod_{j=1}^{d} s(n,l_j).$$

Proof. If $G = K_{n,n,\cdots,n}$ is a complete d-partite graph ,then
$\bar{G} = K_n \cup K_n \cup \cdots \cup K_n$,any p,q,$p = q = n$,$K_p \cap K_q = \phi$.
The chromatic polynomial of $\bar{G}$ is

$$f(\bar{G},t) = f(K_n,t) \cdot f(K_n,t) \cdot \cdots \cdot f(K_n,t)$$

$$f(\bar{G},t) = \prod_{k=1}^{n}(t-k+1)\prod_{k=1}^{n}(t-k+1)\cdots\prod_{k=1}^{n}(t-k+1)$$

$$= \sum_{l_1=0}^{n} s(n,l_1)t^{l_1} \sum_{l_2=0}^{n} s(n,l_2)t^{l_2} \cdots \sum_{l_d=0}^{n} s(n,l_d)t^{l_d}$$

$$= \sum_{r=0}^{dn}[\sum_{l_1+l_2+\cdots+l_d=r} s(n,l_1)s(n,l_2)\cdots s(n,l_d)]t^r,$$

then

$$Y_p = \sum_{\sum_{1\le j\le d} l_j = p} \prod_{j=1}^{d} s(n,l_j), 1 \le p \le dn,$$

where $n+n+\cdots+n = dn$. By Theorem 3.1,then the number of $S^{(n)}-$factors with exactly k components in any complete d-partite graph $K_{n,n,\cdots,n}$

$$N(G,k) = \sum_{p=k}^{dn} N(K_p,k) \sum_{\sum_{1\le j\le d} l_j = p} \prod_{j=1}^{d} s(n,l_j).$$

K_k-factor is a special case of $S^{(n)}-$factor.Because G is a complete d-partite graph,then G is the graph without K_{d+1} subgraphs.

Each component of $S^{(dn)}-$factors with exactly n components in G is all isomorphic to K_d. So that $N(G,n)$ is equal to the number of all K_d-factors in G. Finally,we gain the explicit formula of all K_d-factors in any complete d-partite graph as follows:

$$N(G,n) = \sum_{p=n}^{dn} N(K_p,n) \sum_{\sum_{1\le j\le d} l_j = p} \prod_{j=1}^{d} s(n,l_j).$$

□

Corollary 7.2. *If G is a complete 3-partite graph $K_{n,n,n}$, then the number of all K_3-factors in any complete 3-partite graph is as follows:*

$$\sum_{p=n}^{3n} N(K_p, n) \sum_{\sum_{1\le j\le 3} l_j = p} \prod_{j=1}^{3} s(n, l_j).$$

Proof. Let $d = 3$. Then by Theorem 7.1 we gain the number of all K_3-factors in any complete 3-partite graph is as follows:

$$\sum_{p=n}^{3n} N(K_p, n) \sum_{\sum_{1\le j\le 3} l_j = p} \prod_{j=1}^{3} s(n, l_j).$$

□

Corollary 7.3. *If G is a complete $3-$partite graph $K_{n,n,n}$, then there exists the combinatorial identity*

$$\sum_{k=1}^{3n} k! N(G, k) \binom{3n}{k} = (n!)^3 \binom{3n}{2n, n}^3,$$

where

$$N(G, k) = \sum_{p=k}^{3n} N(K_p, k) \sum_{\sum_{1\le j\le 3} l_j = p} \prod_{j=1}^{3} s(n, l_j).$$

Proof. If $G = (X_1, X_2, X_3)$is a complete $3-$partite graph, $|X_1| = |X_2| = |X_3| = n$, namely $K_{n,n,n}$,then

$$\sum_{k=1}^{3n} k! N(G, k) \binom{3n}{k} = (n!)^3 \binom{3n}{2n, n}^3.$$

(see LiMin Yang and TianMing Wang[12]).By the proving course of Theorem 7.1, we obtain the formula

$$N(G, k) = \sum_{p=k}^{dn} N(K_p, k) \sum_{\sum_{1\le j\le d} l_j = p} \prod_{j=1}^{d} s(n, l_j).$$

Let $d = 3$.Then we derive the combinatorial identity

$$\sum_{k=1}^{3n} k! N(G, k) \binom{3n}{k} = (n!)^3 \binom{3n}{2n, n}^3,$$

where

$$N(G, k) = \sum_{p=k}^{3n} N(K_p, k) \sum_{\sum_{1\le j\le 3} l_j = p} \prod_{j=1}^{3} s(n, l_j).$$

□

Corollary 7.4. *If G is a complete $2-$partite graph $K_{n,n}$,then there exists the combinatorial identity*

$$\sum_{k=1}^{2n} k!N(G,k)\binom{2n}{k} = (n!)^2\binom{2n}{n,n}^2,$$

where

$$N(G,k) = \sum_{p=k}^{2n} N(K_p,k) \sum_{l_1+l_2=p} s(n,l_1)s(n,l_2)$$

Proof. If $G = (X_1, X_2)$is a complete $2-$partite graph ,$|X_1| = |X_2| = n$,namely $K_{n,n}$,then

$$\sum_{k=1}^{2n} k!N(G,k)\binom{2n}{k} = (n!)^2\binom{2n}{n,n}^2.$$

(see LiMin Yang and TianMing Wang[12]).By the proving course of Theorem 7.1,

$$N(G,k) = \sum_{p=k}^{dn} N(K_p,k) \sum_{\sum\limits_{1\le j\le d} l_j=p} \prod_{j=1}^{d} s(n,l_j).$$

Let $d = 2$. Then we give the combinatorial identity

$$\sum_{k=1}^{2n} k!N(G,k)\binom{2n}{k} = (n!)^2\binom{2n}{n,n}^2,$$

where

$$N(G,k) = \sum_{p=k}^{2n} N(K_p,k) \sum_{l_1+l_2=p} s(n,l_1)s(n,l_2)$$

□

Here we will give the counting theorem of $K_l - factors$ in G without any K_{l+1} subgraph(or $K_{l+1}-$free graphs).

Theorem 7.5. *If G is a graph without any K_{l+1}subgraph(or $K_{l+1}-$free graphs) and has n vertices, and the chromatic polynomial of the complementary graph $\bar{G}$ of G is $f(\bar{G},t) = \sum\limits_{p=1}^{n} Y_p t^p$, then the number of all K_l-factors with exactly m components is the following*

$$N(G,m) = \sum_{p=m}^{n} N(K_p,m)Y_p\ ,$$

where $n = ml$.

Proof. Because G is the graph without any K_{l+1} subgraph, Each component of S^n-factors with exactly m components in G is all isomorphic to K_l, and $n = ml$, so that $N(G,m)$ is equal to the number of all K_l-factors in G. By Theorem 3.1, then

$$N(G,m) = \sum_{p=m}^{n} N(K_p,m)Y_p\ ,$$

where $n = ml$. □

8. The counting formula of the number of covers on shortest circles

C_n is a longest circle in any graph G, C_3 is a shortest circle in any graph G, and $K_3 = C_3$.

Definition 8.1. If M is a spanning subgraph of the graph G,each component of M is all isomorphic to C_3, then M is called one cover of shortest circles.

Let $\Omega(G)$ denote the number of all covers of shortest circles.

Theorem 8.2. *If G is a complete 3-partite graph $K_{n,n,n}$, then the number all covers on shortest circles in any complete 3-partite graph is as follows:*

$$\Omega(G) = \sum_{p=n}^{3n} N(K_p, n) \sum_{\sum_{1\leq j\leq 3} l_j = p} \prod_{j=1}^{3} s(n, l_j).$$

Proof. Let $d = 3$.Then by Theorem 7.1 we gain the number of all K_3-factors in any complete 3-partite graph is as follows:

$$\sum_{p=n}^{3n} N(K_p, n) \sum_{\sum_{1\leq j\leq 3} l_j = p} \prod_{j=1}^{3} s(n, l_j).$$

Because of $K_3 = C_3$,then $\Omega(G)$ is equal to the number of all K_3-factors in any complete 3-partite graph G. So that by using of enumeration of $S^{(n)}$-factors , we solve the counting formula of the number of covers on shortest circles as follows:

$$\Omega(G) = \sum_{p=n}^{3n} N(K_p, n) \sum_{\sum_{1\leq j\leq 3} l_j = p} \prod_{j=1}^{3} s(n, l_j).$$

Here we will give the counting theorem of the number of all covers of shortest circles in any graph G without K_4 subgraphs(or K_4-free graph). □

Theorem 8.3. *If G is a graph without K_4 subgraphs(or K_4-free graph) and has n vertices, and the chromatic polynomial of the complementary graph $\bar{G}$ of G is $f(\bar{G}, t) = \sum_{p=1}^{n} Y_p t^p$,then the number of all covers of shortest circles is as follows:*

$$\Omega(G) = \sum_{p=m}^{3m} N(K_p, m) Y_p ,$$

where $n = 3m$.

Proof. Because G is the graph without K_4 subgraphs, each component of

$S^{(n)}-$factors with exactly m components in G is all isomorphic to K_3,and $n = 3m$,so that $N(G, m)$ is equal to the number of all K_3-factors in G. By Theorem 3.1,then

$$N(G, m) = \sum_{p=m}^{3m} N(K_p, m) Y_p ,$$

where $n = 3m$. On the other hand $K_3 = C_3$, the number of all covers of shortest circles is equal to the number of all K_3-factors in G without any K_4 subgraph. Then

$$\Omega(G) = N(G, m).$$

Finally,we derive the counting formula of the number of covers on shortest circles in G without any K_4 subgraph

$$\Omega(G) = \sum_{p=m}^{3m} N(K_p, m) Y_p \ ,$$

where $n = 3m$. □

9. The mean color number of any windgraph K_n^d

Definition 9.1. For any n-coloring Γ of G, let $L(\Gamma)$ denote the actual number of colors used, the average of $L(\Gamma)^{,s}$ over all n-coloring Γ is called the mean color number.

Let $\mu(G)$ denote the mean colour number of any graph.

Theorem 9.2. *If $\mu(G)$ is the mean colour number of any windgraph K_n^d, and*

$$H(t) = \sum_{k=1}^{d(n-1)+1} (d(n-1)+1)_k \sum_{p=k-1}^{d(n-1))} N(K_p, k-1) \sum_{\sum_{1\le j\le d} l_j = p} \prod_{j=1}^{d} s(n-1, l_j) t^k \ ,$$

then the explicit formula

$$\mu(K_n^d) = \frac{H'(1)}{H(1)} \ .$$

Proof. If $\mu(G)$ is the mean colour number of any windgraph K_n^d, then $\bar{G} = K_1 \bigcup H$ and $K_1 \cap H = \phi$, $H$is a complete d-partite graph, let$H = K_{n-1,n-1,\cdots,n-1}$, the number of $n-1$ is d ,we have the formula

$$N(\bar{G}, k) = \sum_{l+m=k} N(K_1, l) N(H, m) = N(H, k-1).$$

By the proving course of Theorem 7.1, we gain

$$N(H, k-1) = \sum_{p=k-1}^{d(n-1)} N(K_p, k-1) \sum_{\sum_{1\le j\le d} l_j = p} \prod_{j=1}^{d} s(n-1, l_j).$$

*If $\mu(G)$ is the mean colour number of any graph with n vertices G,*and

$$H(t) = \sum_{k=1}^{n} (n)_k N(\bar{G}, k) t^k \ ,$$

then

$$\mu(G) = \frac{H'(1)}{H(1)} \ .$$

(see Yang and Wang[17],2004).Here we have solved the counting formula of $N(\bar{G},k)$ by the representing formula of $N(G,k)$. Finally, we derive the explicit formula

$$\mu(K_n^d)=\frac{H'(1)}{H(1)},$$

where

$$H(t)=\sum_{k=1}^{d(n-1)+1}(d(n-1)+1)_k\sum_{p=k-1}^{d(n-1)}N(K_p,k-1)\sum_{\sum_{1\leq j\leq d}l_j=p}\prod_{j=1}^{d}s(n-1,l_j)t^k\ .$$

For any windgraph graph,by means of methods of Bartels and Welsh(1995)

(see[18]), and F.M.Dong(2003)(see[19]),we can't gain the explicit formula. By the representing formula of $N(G,k)$, we gain a new way to obtain the explicit formulas on the mean color numbers. □

10. Conclusions and future work

In this paper, our main problems are the representing formula of $N(G,k)$ and the counting formula of K_d-factors in any complete d-partite graph $K_{n,n,\cdots,n}$,we have solved the representing formula of $N(G,k)$ and the counting formula of complete d-partite graph.We present classes of graphs counting formulas for $N(G,k)$ and $A(G)$, specially, gain the explicit formula of $N(\bar{T},k)$ of the complementary $\bar{T}$ of any tree T, and derive some new combinatorial identities.Finally, we solve the counting formula of the number of covers on shortest circles.The representing formula of $N(G,k)$ has be cited in Yang and Wang [16] and [17],2004,India.We have cited the representing formula of $N(G,k)$,Reviewer Cédric Lamthe ,2005.In future work,the representing formula of $N(G,k)$ will be cited again by us in our papers, we will solve independent set polynomials $I(G,x)$.

Independent set polynomials $I(G,x)$ are defined as

$$I(G,x)=\sum_{k=1}^{n}b_k(G)x^k=\sum_{I\subset v(G)}\prod_{v\in I}x,$$

where let $b_k(G)$ be exactly k-independent sets of G .

Complexity: It is easy to see that $I(G,x)$ is NP-hard to compute.(see [21]).

The representing formula of $N(G,k)$ will be gained deep applications in future research.

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