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HOAOYA INDEX AND THE MATCHING DEFECT POLYNOMIALS OF A GRAPH G WITHOUT K_3 SUBGRAPH

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ABSTRACT. For a simple graph G without K_3 subgraph, by means of discussing a method of 1-1 mapping and a method of complete components, the author proves that its k -matching number ϕ_k with its the number $N(G, n-k)$ of $S^{(n)}$ -factors with exactly $(n-k)$ components is equal, and its the Hosoya index $Z(G)$ with its the number $A(G)$ of $S^{(n)}$ -factors is equal through analyzing the relations k -matching of a simple graph G without K_3 subgraph with exactly $(n-k)$ components; Finally, the author solves the recurrence relation of the number ϕ_{k, T_t} of k -matching of the regular m -furcating tree, so gets its the matching defect polynomials $m(T, x)$; and a recurrence formula of the Hosoya index Z_t of the regular m -furcating tree; and a counting formula of the Hosoya index Z_G of the graph by q chord graph of cycle. Simultaneously the corresponding examples are given.

1. Basic introduction

In the mathematical discipline of graph theory, a matching or independent edge set in a graph is a set of edges without common vertices. It may also be an entire graph consisting of edges without common vertices.

Given a graph $G = (V, E)$, a matching M in G is a set of pairwise non-adjacent edges; that is, no two edges share a common vertex.

A vertex is matched (or saturated) if it is incident to an edge in the matching. Otherwise the vertex is unmatched. For example:(see Figure 1-1), the edge subsets $\{ab, gh, ef\}$, $\{cg, de\}$, $\{ab\}$ and the empty set were one of its matching.

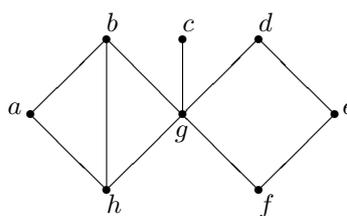


Figure 1-1

If M is subset with k edges in G , $|M| = k$, then M is called one k -matching of G . The number of k -matching, is called one k -matching number of G .

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Definition 1.1. The Hosoya index, also known as the Z index, of a graph is the total number of matchings in it. Namely:

$$Z(G) = \sum_{k=0}^{\lfloor n/2 \rfloor} \phi_k,$$

where ϕ_k is the k -matching number of graph G , denoted $\phi_0 = 1$.

The Hosoya index is always at least one, because the empty set of edges is counted as a matching for this purpose. Equivalently, the Hosoya index is the number of non-empty matchings plus one.

This graph invariant was introduced by Haruo Hosoya in 1971. It is often used in chemoinformatics for investigations of organic compounds.

In his article "The Topological Index Z Before and After 1971" on the history of the notion and the associated inside stories, Hosoya writes that he introduced the Z index to report a good correlation of the boiling points of alkane isomers and their Z indices, basing on his unpublished 1957 work carried out while he was an undergraduate student at the University of Tokyo.

For example: see Figure 1-2, Hosoya index the application of the linear alkane.

A linear alkane, for the purposes of the Hosoya index, may be represented as a path graph without any branching. A path with one vertex and no edges (corresponding to the methane molecule) has one (empty) matching, so its Hosoya index is one; a path with one edge (ethane) has two matchings (one with zero edges and one with one edges), so its Hosoya index is two. Propane (a length-two path) has three matchings: either of its edges, or the empty matching. n -butane (a length-three path) has five matchings, distinguishing it from isobutane which has four. More generally, a matching in a path with k edges either forms a matching in the first $k-1$ edges, or it forms a matching in the first $k-2$ edges together with the final edge of the path.

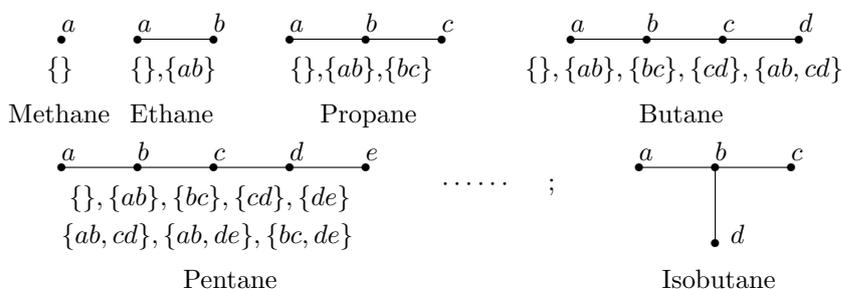


Figure 1-2

Note:1) The difference between butane isomers iso-butane, isobutane is non-linear alkanes, it has only four matches. Similarly, right after the linear alkanes butane and its isomers have done such a distinction.

2)The Hosoya indices of linear alkanes obey the recurrence governing the Fibonacci Sequence. This structure of the matchings in these graphs may be visualized using a Fibonacci cube. Here we are given the chemical interpretation of Fibonacci Sequence.

Hosoya index since being introduced, there have been rich in results:

- (see[1]) The bound for Hosoya topological index of tree and single-cycle graph. Some methods in combinatorics are used. A recursion arithmetic method is given to compute the Hosoya index of a tree.
- (see[2]) Hosoya Index Sequence of a Class of Trees.
- (see[3]) Ordering Unicyclic Graphs with Respect to Hosoya Index. its investigate the Hosoya index of unicyclic graphs and get the first eighth smallest unicyclic graphs with respect to the Hosoya index.
- (see[4]) The orderings of trees with respect to Hosoya Index.
- (see[5]) Hosoya index extreme on some specific trees. The tree with maximal Hosoya index in graph $H_{n,d,3}$ is discussed in this paper.

2. Basic Knowledge

In the mathematical field of graph theory, a complete graph is a simple graph in which every pair of distinct vertices is connected by an edge. The complete graph on n vertices has n vertices and $n(n - 1)/2$ edges, and is denoted by K_n

For example: see Figure 2-1.

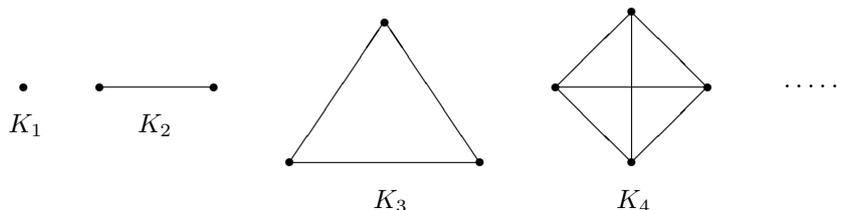


Figure 2-1

A subgraph of a graph G is a graph whose vertex set is a subset of that of G , and whose adjacency relation is a subset of that of G restricted to this subset. We say a graph G contains another graph H if some subgraph of G is H or is isomorphic to H .

A subgraph H is a spanning subgraph, or factor, of a graph G if it has the same vertex set as G . We say H spans G .

A subgraph H of a graph G is said to be induced if, for any pair of vertices x and y of H , xy is an edge of H if and only if xy is an edge of G . In other words, H is an induced subgraph of G if it has all the edges that appear in G over the same vertex set. If the vertex set of H is the subset D of $V(G)$, then H can be written as $G[D]$ and is said to be induced by D .

In graph theory, a connected component of an undirected graph is a subgraph in which any two vertices are connected to each other by paths, and to which no more vertices or edges can be added while preserving its connectivity. That is, it is a maximal connected subgraph. A graph that is itself connected has exactly one connected component, consisting of the whole graph.

In graph theory, an isomorphism of graphs G and F is a bijection between the vertex sets of G and F

$$\Phi : V(G) \rightarrow V(F), \quad \Psi : E(G) \rightarrow E(F),$$

such that any two vertices u and v of G are adjacent in G if and only if $\Phi(u)$ and $\Phi(v)$ are adjacent in F

$$\Psi(uv) = \Phi(u)\Phi(v).$$

If an isomorphism exists between two graphs, then the graphs are called isomorphic and we write $G \cong F$. For example: see Figure 2-2

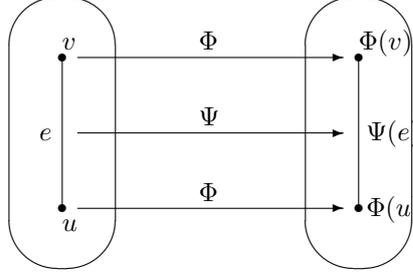


Figure 2-2

Definition 2.1 ([6]). For $S^{(n)} = \{K_i : 1 \leq i \leq n\}$, $n \geq 1$, K_i is a complete graph with i vertices, if M is a subgraph of a any graph G , and each component of M is all isomorphic to some element of $S^{(n)} = \{K_i : 1 \leq i \leq n\}$, then M is called one $S^{(n)}$ -subgraph, if M is a spanning subgraph of G , then M is called one $S^{(n)}$ -factor of G .

Let $N(G, k)$ denote the number of $S^{(n)}$ -factors with exactly k components.

Definition 2.2 ([6]). In the root of tree T , if any vertex out-degree at most m , then T is called one m -furcating tree; A out-degree of non-zero vertex is called branching vertex, each branch vertex out-degree both equal to m , then T is called the complete m -furcating tree; The vertex of zero out-degree is called leaf,if all the leaves vertex at the same level, then T is called the regular m -furcating tree.

Definition 2.3 ([7]). A k -matching in a graph G is a set of k edges, no two of which have a vertex in common. Let ϕ_k be the number of k -matchings of the graph G , with $\phi_0(G) = 1$ and $\phi_1(G) = m$ is the number of edges of G . Then the matching polynomial is defined by

$$m(G, x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \phi_k(G) x^{n-2k},$$

where n is the number of vertices of G .

3. Basic Lemmas

T is a regular m -furcating tree, i is the number of branching vertices, t is the number of leaves, let $N(T, k)$ denote the number of $S^{(n)}$ -factors with exactly k components of the regular m -furcating tree with the number t of leaves. $A(T)$ is the number of $S^{(n)}$ -factors of the regular m -furcating tree, namely, $A(G) =$

$$\sum_{k=1}^n N(T, k).$$

Lemma 3.1 ([8]). *If T is a regular m -furcating tree, and i is the branching vertices, and t is the number of leaves, $t \geq m^3$, for example: Figure 3-1. then*

$$\begin{aligned}
 N(T_t, k) &= \sum_{\substack{\sum_{1 \leq j \leq m} l_j = k-1}} N(T_{t/m}, l_1) \cdots N(T_{t/m}, l_m) \\
 &+ m \sum_{\substack{\sum_{1 \leq j \leq 2m-1} l_j = k-1}} N(T_{t/m^2}, l_1) \cdots N(T_{t/m^2}, l_m) \cdot N(T_{t/m}, l_{m+1}) \\
 &\cdots N(T_{t/m}, l_{2m-1})
 \end{aligned}$$

The regular m -furcating tree, t is the number of leaves, $t \geq m^3$.

Initial Value:

$$N(T_2, k) = \begin{cases} 0 & k = 1, \\ 2 & k = 2, \\ 1 & k = 3. \end{cases} \quad N(T_4, k) = \begin{cases} 0 & 1 \leq k \leq 4, \\ 8 & k = 5, \\ 6 & k = 6, \\ 1 & k = 7. \end{cases}$$

Lemma 3.2 ([6]). *If T is a regular m -furcating tree, and i is the branching vertices, and t is the number of leaves, for example: Figure 3-1. then*

$$A_t = A_{t/m}^m + mA_{t/m^2}^m A_{t/m}^{m-1},$$

Initial Value:

$$A_m = m + 1, \quad A_{m^2} = (2m + 1)(m + 1)^{m+1}.$$

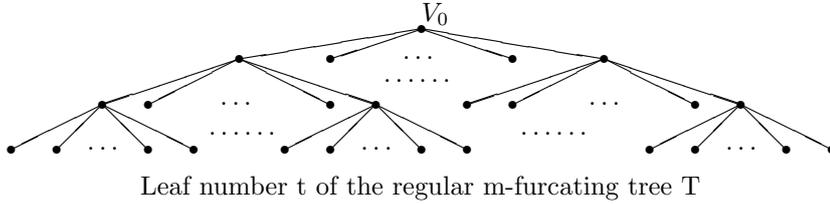


Figure 3-1

Lemma 3.3 ([9]). *There are n vertices of the circle C_n on the number of $S^{(n)} = \{K_i : 1 \leq i \leq n\}$ -factors*

$$A(C_n) = a^n + b^n = L_n, \quad n \geq 4, a = \frac{1 - \sqrt{5}}{2}, b = \frac{1 + \sqrt{5}}{2},$$

where L_n is Lucas sequence the n -th item.

Lemma 3.4 ([9]). *There are $n + 1$ vertices formed by a length n path P_n , its the number of $S^{(n)} = \{K_i : 1 \leq i \leq n\}$ -factors*

$$A(P_n) = \frac{b^{n+2} - a^{n+2}}{\sqrt{5}} = F_{n+1}, \quad n \geq 0, a = \frac{1 - \sqrt{5}}{2}, b = \frac{1 + \sqrt{5}}{2},$$

where F_{n+1} is Fibonacci sequence the first $n + 1$ -th items.

Lemma 3.5. *In the non- K_3 subgraph graph G with n vertices, if there is a k -matching, then it corresponds to an $S^{(n)} = \{K_i : 1 \leq i \leq n\}$ -factor with exactly $(n - k)$ components; Contrary, if there is an $S^{(n)} = \{K_i : 1 \leq i \leq n\}$ -factor with exactly $(n - k)$ components, then it corresponds to a k -matching.*

Proof. When $k \leq \lfloor n/2 \rfloor$, G is a graph with n vertices without K_3 subgraph, so its $S^{(n)} = \{K_i : 1 \leq i \leq n\}$ -factors that only K_1 and K_2 , and because there are k matching graph G , so it's $S^{(n)} = \{K_i : 1 \leq i \leq n\}$ -factors were k K_2 , the formation of k branches, while the remaining $(n - 2k)$ vertices, namely, $(n - 2k)$ a K_1 , the formation of $(n - 2k)$ branches, so it's $S^{(n)} = \{K_i : 1 \leq i \leq n\}$ -factor in a total of $(n - k)$ branches;

Contrary, because the G is a graph with n vertices without K_3 subgraph, so it's $S^{(n)} = \{K_i : 1 \leq i \leq n\}$ -factor that only K_1 and K_2 , it's $S^{(n)} = \{K_i : 1 \leq i \leq n\}$ -factors in a total of $(n - k)$ branches, that is, the total number of K_1 and K_2 is equal to $(n - k)$. suppose K_1 have a and K_2 have b , by

$$\begin{cases} a + b = n - k \\ a + 2b = n \end{cases}$$

so that we have the equal systems

$$\begin{cases} b = k \\ a = n - 2k \end{cases}$$

namely: $K_1 = n - 2k$, $K_2 = k$, that is, there are k matching or a k -matching in the graph.

When $k > \lfloor n/2 \rfloor$, such a k -matching does not exist; similarly, at this time the number of components in the $S^{(n)} = \{K_i : 1 \leq i \leq n\}$ -factor is

$$n - k < n - \lfloor n/2 \rfloor = \begin{cases} \lfloor n/2 \rfloor & \text{When the } n \text{ is an even,} \\ \lfloor n/2 \rfloor + 1 & \text{When the } n \text{ is odd.} \end{cases}$$

And because graph G only K_1 and K_2 , so the number of components a minimum of $\lfloor n/2 \rfloor$ or $\lfloor n/2 \rfloor + 1$, contradictions, that such a number of components of $S^{(n)} = \{K_i : 1 \leq i \leq n\}$ -factor does not exist. \square

Note: For example, See Figure 3-2 in page 17th. In the graph G without K_3 subgraph with n vertices, when the d_{2k-1} is not with the d_{2k} constitute a match in the k -matching, but with the d_{2k+1} constitute a match, Then v_{2k-1} and v_{2k} are not associated, but with v_{2k+1} associated in the $S^{(n)} = \{K_i : 1 \leq i \leq n\}$ -factor, namely: $v_{2k-1}v_{2k+1}$ constitute a K_2 , v_{2k} form a K_1 . And vice versa.

Lemma 3.6. *In the non- K_3 subgraph graph G with n vertices, for any a k -matching exactly the only one there are $(n-k)$ components of the $S^{(n)} = \{K_i : 1 \leq i \leq n\}$ -factor with its counterpart; On the contrary, for any one has $(n-k)$ exactly components of the $S^{(n)} = \{K_i : 1 \leq i \leq n\}$ -factor there are also the only one k -matching and its counterpart, namely: it is 1-1 mapping relationship between k -matching and $(n-k)$ exactly one branch of the $S^{(n)}$ -factor, such that*

$$f : k\text{-matching} \leftrightarrow S^{(n)\text{-factors with exactly } (n - k) \text{ components}}$$

Proof. Omitted. \square

4. Main Theorems

Theorem 4.1. *In the non- K_3 subgraph graph G with n vertices, its the number of k -matching with its the number of $S^{(n)}$ -factors with exactly $(n - k)$ components is equal, namely*

$$\phi_k = N(G, n - k).$$

Proof. By Lemma 3.6, so its k -matching and $S^{(n)}$ -factors with exactly $(n - k)$ components is equal to the base, namely $\phi_k = N(G, n - k)$. \square

Theorem 4.2. *In the non- K_3 subgraph graph G with n vertices, the total number of k -matching, namely Hosoya index $Z(G) = \sum_{k=0}^{\lfloor n/2 \rfloor} \phi_k$ with its the number of $S^{(n)}$ -factor is equal:*

$$Z(G) = \sum_{k=0}^{\lfloor n/2 \rfloor} \phi_k = A(G).$$

Proof. By the course of Lemma 3.5,

When $k > \lfloor n/2 \rfloor$

$$N(G, n - k) = 0,$$

then

$$\sum_{k=\lfloor n/2 \rfloor+1}^n N(G, n - k) = 0;$$

by Theorem 4.1,

$$\phi_k = N(G, n - k)$$

then

$$\begin{aligned} Z(G) &= \sum_{k=0}^{\lfloor n/2 \rfloor} \phi_k = \sum_{k=0}^{\lfloor n/2 \rfloor} N(G, n - k) \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} N(G, n - k) + \sum_{k=\lfloor n/2 \rfloor+1}^n N(G, n - k) \\ &= \sum_{k=0}^n N(G, n - k) \\ &= \sum_{k=0}^n N(G, k) \\ &= A(G). \end{aligned}$$

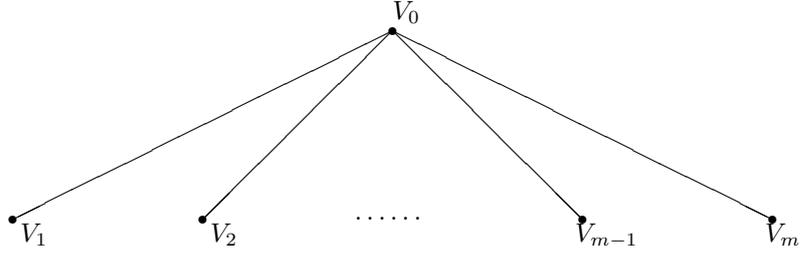
\square

Example 1: see Figure 4-1, If G is a Star graph $K_{1,m}$, seeking its Hosoya index $Z(G) = ?$

Answers: By Figure 4-1, $\phi_0 = 1, \phi_1 = m$, so Star graph $K_{1,m}$ of the Hosoya index is

$$Z(G) = \phi_0 + \phi_1 = m + 1.$$

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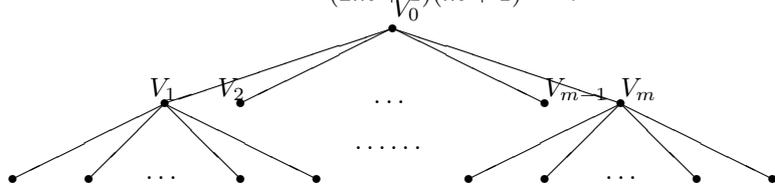
Star Graph

Figure 4-1

Example 2: see Figure 4-2, if T is the regular m -furcating tree, branch vertices for the $m + 1$. Seeking its Hosoya index $Z(T) = ?$

Answers: By Figure 4-2,

$$\begin{aligned} Z(T - V(K_1)) &= (m + 1)^m \\ Z(T - V(K_2)) &= m(m + 1)^{m-1}. \\ Z(T) &= Z(T - V(K_1)) + Z(T - V(K_2)) \\ &= (m + 1)^m + m(m + 1)^{m-1} \\ &= (2m + 1)(m + 1)^{m-1}. \end{aligned}$$



the regular m -furcating tree with $m + 1$ branch vertices

Figure 4-2

5. Applications

Let ϕ_{k,T_t} denote the number of k -matching of a regular m -furcating tree that leaf number is t ; $m(T, x)$ is the matching defect polynomial of a regular m -furcating tree; Z_t is Hosoya index of a regular m -furcating tree.

Theorem 5.1. *If T is a regular m -furcating tree, i is the the number of branching vertices, t is the number of leaves, $n = i + t$, $t \geq m^3$, for example, see Figure 3-2, then its k -matching number:*

$$\begin{aligned} \phi_{k,T_t} &= \sum_{\sum_{1 \leq j \leq m} l_j = n-k-1} N(T_{t/m}, l_1) \cdots N(T_{t/m}, l_m) \\ &+ m \sum_{\sum_{1 \leq j \leq 2m-1} l_j = n-k-1} N(T_{t/m^2}, l_1) \cdots N(T_{t/m^2}, l_m) \cdot N(T_{t/m}, l_{m+1}) \\ &\cdots N(T_{t/m}, l_{2m-1}). \end{aligned}$$

Initial values:

$$N(T_2, k) = \begin{cases} 0 & \text{when } k = 1, \\ 2 & \text{when } k = 2, \\ 1 & \text{when } k = 3. \end{cases} \quad N(T_4, k) = \begin{cases} 0 & \text{when } 1 \leq k \leq 4, \\ 8 & \text{when } k = 5, \\ 6 & \text{when } k = 6, \\ 1 & \text{when } k = 7. \end{cases}$$

Proof. By Lemma 3.1

$$\begin{aligned} N(T_t, k) &= \sum_{\substack{\sum_{1 \leq j \leq m} l_j = k-1}} N(T_{t/m}, l_1) \cdots N(T_{t/m}, l_m) \\ &\quad + m \sum_{\substack{\sum_{1 \leq j \leq 2m-1} l_j = k-1}} N(T_{t/m^2}, l_1) \cdots N(T_{t/m^2}, l_m) \cdot N(T_{t/m}, l_{m+1}) \\ &\quad \cdots N(T_{t/m}, l_{2m-1}) \end{aligned}$$

we using $n - k$ replace k ,

$$\begin{aligned} N(T_t, n - k) &= \sum_{\substack{\sum_{1 \leq j \leq m} l_j = n-k-1}} N(T_{t/m}, l_1) \cdots N(T_{t/m}, l_m) \\ &\quad + m \sum_{\substack{\sum_{1 \leq j \leq 2m-1} l_j = n-k-1}} N(T_{t/m^2}, l_1) \cdots N(T_{t/m^2}, l_m) \\ &\quad \cdot N(T_{t/m}, l_{m+1}) \cdots N(T_{t/m}, l_{2m-1}) \end{aligned}$$

by Theorem 4.1 $\phi_k = N(G, n - k)$, then

$$\begin{aligned} \phi_{k, T_t} &= \sum_{\substack{\sum_{1 \leq j \leq m} l_j = n-k-1}} N(T_{t/m}, l_1) \cdots N(T_{t/m}, l_m) \\ &\quad + m \sum_{\substack{\sum_{1 \leq j \leq 2m-1} l_j = n-k-1}} N(T_{t/m^2}, l_1) \cdots N(T_{t/m^2}, l_m) \cdot N(T_{t/m}, l_{m+1}) \\ &\quad \cdots N(T_{t/m}, l_{2m-1}). \end{aligned}$$

Initial values:

$$N(T_2, k) = \begin{cases} 0 & \text{when } k = 1, \\ 2 & \text{when } k = 2, \\ 1 & \text{when } k = 3. \end{cases} \quad N(T_4, k) = \begin{cases} 0 & \text{when } 1 \leq k \leq 4, \\ 8 & \text{when } k = 5, \\ 6 & \text{when } k = 6, \\ 1 & \text{when } k = 7. \end{cases}$$

□

Corollary 5.2. *If T is a regular 2-furcating tree, i is the the number of branching vertices, t is the number of leaves, $n = i + t$, $t \geq 8$, then its k -matching number:*

$$\begin{aligned} \phi_{k, T_t} &= \sum_{l_1 + l_2 = n - k - 1} N(T_{t/2}, l_1) N(T_{t/2}, l_2) \\ &\quad + 2 \sum_{l_1 + l_2 + l_3 = n - k - 1} N(T_{t/4}, l_1) N(T_{t/4}, l_2) N(T_{t/2}, l_3). \end{aligned}$$

Proof. Omitted □

Theorem 5.3. *The matching defect polynomial of a regular m -furcating tree is*

$$m(T, x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \phi_k(T) x^{n-2k},$$

where $\phi_k(T)$ is the number of k -matching of the regular m -furcating tree.

Proof. Omitted □

Example 1: see Figure 4-2, when $t = 8$, seek the number of k -matching ϕ_k and the matching defect polynomials of the regular 2-furcating tree?

Answers: known $t = 8$, $m = 2$, then $i = 7$, $n = 15$.

$$\begin{aligned} \phi_{k, T_8} &= N(T_8, 15 - k) \\ &= \sum_{l_1+l_2=14-k} N(T_4, l_1)N(T_4, l_2) + 2 \sum_{l_1+l_2+l_3=14-k} N(T_2, l_1) \\ &\quad N(T_2, l_2) \cdot N(T_4, l_3). \\ \phi_{0, T_8} &= N(T_8, 15) \\ &= \sum_{l_1+l_2=14} N(T_4, l_1)N(T_4, l_2) + 2 \sum_{l_1+l_2+l_3=14} N(T_2, l_1) \\ &\quad N(T_2, l_2) \cdot N(T_4, l_3) \\ &= N(T_4, 7) \cdot N(T_4, 7) \\ &= 1 \\ \phi_{1, T_8} &= N(T_8, 14) \\ &= \sum_{l_1+l_2=13} N(T_4, l_1) \cdot N(T_4, l_2) + 2 \sum_{l_1+l_2+l_3=13} N(T_2, l_1) \\ &\quad \cdot N(T_2, l_2) \cdot N(T_4, l_3) \\ &= 2 \cdot N(T_4, 7) \cdot N(T_4, 6) + 2 \cdot N(T_2, 3) \cdot N(T_2, 3) \cdot N(T_4, 7) \\ &= 2 \times 6 + 2 \times 1 \times 1 \times 1 \\ &= 14 \\ \phi_{2, T_8} &= 72, \phi_{3, T_8} = 168, \phi_{4, T_8} = 176, \phi_{5, T_8} = 64. \end{aligned}$$

The matching defect polynomials is

$$\begin{aligned} m(T, x) &= \sum_{k=0}^7 (-1)^k \phi_k(T_8) x^{15-2k} \\ &= x^{15} - 14x^{13} + 72x^{11} - 168x^9 + 176x^7 - 64x^5. \end{aligned}$$

Theorem 5.4. *If T is a regular m -furcating tree, i is the the number of branching vertices, t is the number of leaves, $t \geq m^3$, then its Hosoya index:*

$$Z(T) = Z_t = Z_{t/m}^m + mZ_{t/m^2}^m \cdot Z_{t/m}^{m-1}$$

Initial values:

$$Z_m = m + 1, Z_{m^2} = (2m + 1)(m + 1)^{m-1}.$$

Proof. Because the regular m -furcating tree is a graph of non- K_3 subgraph, by Theorem 4.2 and Lemma 3.2,

$$Z(T) = A_t = A_{t/m}^m + mA_{t/m^2}^m \cdot A_{t/m}^{m-1}$$

according to above as follows:

$$Z(T) = Z_t = Z_{t/m}^m + mZ_{t/m^2}^m \cdot Z_{t/m}^{m-1}.$$

□

Corollary 5.5. *If T is a regular m -furcating tree, i is the the number of branching vertices, t is the number of leaves, $t \geq 8$, then its Hosoya index:*

$$Z_t = Z_{t/2}^2 + 2Z_{t/4}^2 \cdot Z_{t/2}.$$

Initial values:

$$Z_2 = 3, Z_4 = 15.$$

Proof. Omitted.

□

Example 2: T is the regular 2-furcating tree, the the number of branching vertices is 8, for example: see Figure 5-1. Seek the Hosoya index $Z_8 = ?$

Answers:By Corollary 5.5,

$$\begin{aligned} Z_8 &= Z_{8/2}^2 + 2Z_{8/4}^2 \cdot Z_{8/2} \\ &= Z_4^2 + 2Z_2^2 \cdot Z_4 \\ &= 15^2 + 2 \times 3^2 \times 15 \\ &= 495 \end{aligned}$$

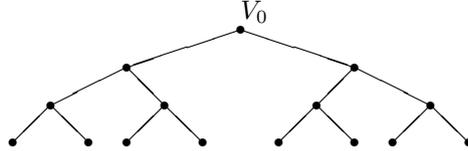


Figure 5-1

The following is some of the initial value table of Hosoya index of a regular 2-furcating tree T

t	2	4	8	16	32
Z_t	3	15	495	467775	44804658375

Theorem 5.6. *The graph H with $n + 1$ vertices, is composed of q chord graph of cycle with n vertices as Figure 5-2. graph G as Figure 5-3(see[9]),*

$H_1 \cap H_2 \cap \dots \cap H_q = K_1$, denoted by O , then its Hosoya index:

$$Z(G) = (L_n)^q + qF_{n-1}(L_n)^{q-1}.$$

where L_n is the n -th item in Lucas numbers, F_{n-1} is the $(n-1)$ -th item in Fibonacci sequence.

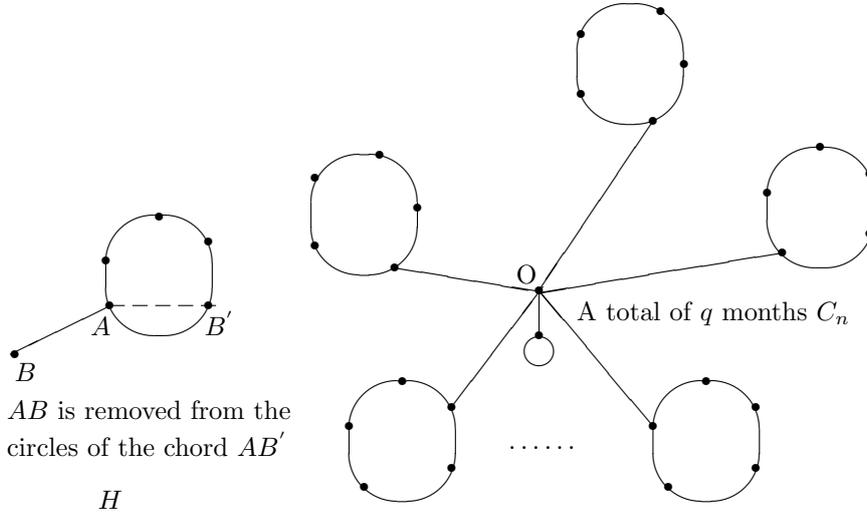


Figure 5-2

Figure 5-3

Proof. By Figure 5-3, because G is a graph of non- K_3 subgraph, and its the number of $S^{(n)} = \{K_i : 1 \leq i \leq n\}$ -factors

$$A(G) = (L_n)^q + qF_{n-1}(L_n)^{q-1}, \text{ (see [9])},$$

by Theorem 4.2,

$$Z_G = A(G) = (L_n)^q + qF_{n-1}(L_n)^{q-1}.$$

□

Example 5: see Figure 5-4, it is composed of 3 chord graph of cycle, Seek the Hosoya index $Z_G = ?$

Answers: Because

$$\begin{aligned} L_4 &= a^4 + b^4 \\ &= \left(\frac{1-\sqrt{5}}{2}\right)^4 + \left(\frac{1+\sqrt{5}}{2}\right)^4 \\ &= 7 \\ F_3 &= \frac{b^4 - a^4}{\sqrt{5}} \\ &= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^4 - \left(\frac{1-\sqrt{5}}{2}\right)^4}{\sqrt{5}} \\ &= 3 \end{aligned}$$

then

$$\begin{aligned} Z_G &= (L_4)^3 + 3 \cdot F_3 \cdot (L_4)^2 \\ &= 7^3 + 3 \times 3 \times 7^2 \\ &= 784 \end{aligned}$$

that is, its Hosoya index is 784.

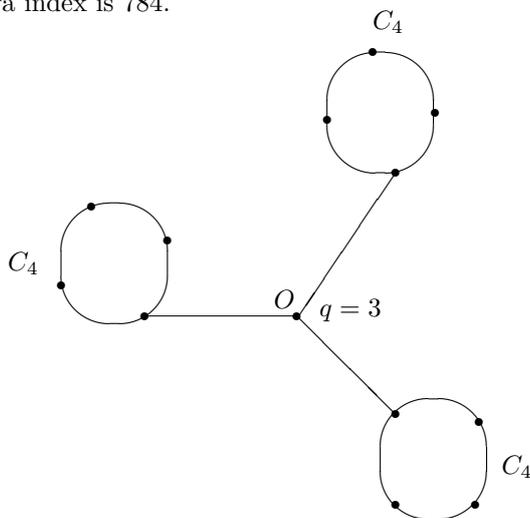


Figure 5-4

6. Conclusions

In this paper, for a any graph G of non- K_3 subgraph, the author set up the relationship of 1-1 mapping between its k -matching and $S^{(n)}$ -factors with exactly $(n - k)$ components, thus proving the expected conclusion.

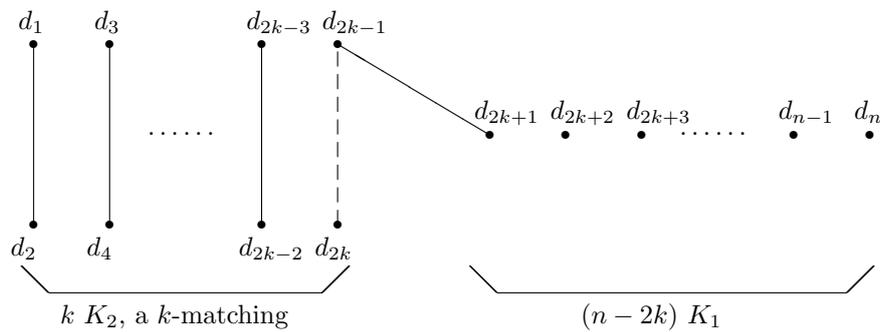
The regular m -furcating tree and the graph by q chord graph of cycle are very representative of the two types of graphs. In the graph theory there is a very important research value, in real life, there is also important, for example: the regular m -furcating tree in the computer, there is significant value; the graph by q chord graph of cycle in the chemical field of organic chemistry is researched more. And so on.

Acknowledgments

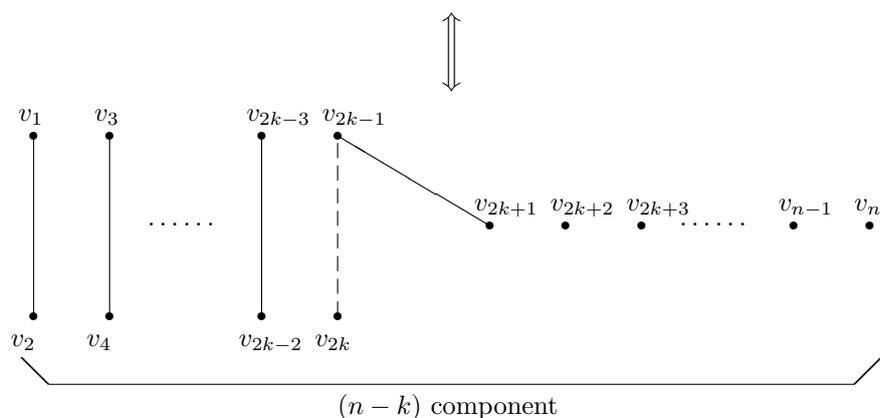
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the k -matching of graph G contains n vertices



the graph G with exactly $(n - k)$ components

Figure 3-2

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