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## EXPLICIT FORMULAS OF PARTITIONS OF STABLE SETS OF THE DIRECT PRODUCT ON SEVERAL PAIRS OF COMPLEMENTARY GRAPHS

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ABSTRACT. In this paper, the main problem is counting of partitions of stable sets of the direct product on several pairs of complementary graphs. Counting of stable set partitions is NP-hard problem. It is difficult the same as counting of crossing number. By means of counting theory of  $S^{(n)}$ -factors and two counting formulas of partitions of stable sets of the direct product graphs, the author obtains explicit formulas of the direct product on several pairs of complementary graphs, such as the wheel graph, the product of  $(n-2)$ -regular graph, etc, and discovers these explicit formulas related to Fibonacci number, Lucas number, Stirling number and Bell number. So that these combinatorial numbers are given graphic interpretations. Further, the product of  $n$  continuous Fibonacci numbers  $F_1 F_2 \dots F_n$  and the product of  $n$  continuous Bell numbers  $b(1)b(2)\dots b(n)$  are given graphic interpretations. Finally, a counter example of the formula for calculating the sum of stable set partitions is presented. The author achieves some advances in its field. For combinatorics and graph theory, it is of reference value and significant.

### 1. Introduction

In [1], the author derives two representation formulas of partitions of stable sets of graphs and relative equalities of partitions of stable sets of the direct product of graphs. In this paper, the author will further discuss explicit formulae of partitions of stable sets of the direct product on several pairs of complementary graphs.

**Definition 1.1** In graph theory, stable set ( or independent set ) is the set of vertices in a graph, no two of which are adjacent. That is, it is a set  $S$  of vertices such that for every two vertices in  $S$ , there is no edge connecting the two.

$\beta(G, k)$  denoted by the number of partitions of  $V(G)$  into exactly  $k$  non-empty stable sets of  $G$ ,  $\beta(G)$  is the number of all partitions of  $V(G)$  into non-empty stable sets of  $G$ .

**Definition 1.2** Let  $S^{(n)} = \{K_i : 1 \leq i \leq n\}$ ,  $n \geq 1$ ,  $K_i$  be a complete graph with  $i$  vertices, if  $M$  is a subgraph of any graph  $G$ , and each component of  $M$  is all isomorphic to some element of  $S^{(n)} = \{K_i : 1 \leq i \leq n\}$ , then  $M$  is called one

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$S^{(n)}$ -subgraph, if  $M$  is a spanning subgraph of  $G$ , then  $M$  is called one  $S^{(n)}$ -factor of  $G$ .

Let  $N(G, k)$  denote the number of  $S^{(n)}$ -factors with exactly  $k$  components.  $A(G)$  is the number of all  $S^{(n)}$ -factors.

**Definition 1.3** The direct product of graphs  $G_1, G_2, \dots, G_n$  denoted by  $G_1 \times G_2 \times \dots \times G_n$ , defined by as follows:

the set of its vertices is  $V(G_1) \cup V(G_2) \cup \dots \cup V(G_n)$

the set of its edges is

$$\{(u, v) | u \in G_i, v \in G_j, i \neq j, 1 \leq i, j \leq n\} \cup E(G_1) \cup E(G_2) \cup \dots \cup E(G_n).$$

## 2. Basic Lemmas

In the section, the author states some basic Lemmas used in the article.

**Lemma 2.1** ([1]) If the chromatic polynomial of any graph  $G$  is  $f(G, t) = \sum_{p=1}^n Y_p t^p$ , then the number of partitions of  $V(G)$  into exactly  $k$  non-empty stable sets of  $G$  is as follows

$$\beta(G, k) = \sum_{p=k}^n S(p, k) Y_p.$$

**Lemma 2.2** ([1]) If  $T$  is a tree with  $n$  vertices, then

$$\beta(T, k) = S(n-1, k-1),$$

where  $S(n-1, k-1)$  is the Stirling number of the second kind.

Graphic interpretation: the Stirling number of the second kind  $S(n, k)$  is the number with exactly  $k+1$  partitions of stable sets in tree  $T$  with  $n+1$  vertices.

**Lemma 2.3** If  $G$  is a cycle  $C_n$ , then

$$\beta(C_n, 1) = 0$$

$$\beta(C_n, k) = \sum_{p=k}^n (-1)^{n-p} \binom{n}{p} S(p, k),$$

$$2 \leq k \leq n.$$

**Proof** The chromatic polynomial of a cycle  $C_n$

$$\begin{aligned} f(G, t) &= (t-1)^n + (-1)^n (t-1) \\ &= \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} t^k + (-1)^n t + (-1)^{n+1} \\ &= (-1)^n + (-1)^{n-1} \binom{n}{1} t + \sum_{k=2}^n (-1)^{n-k} \binom{n}{k} t^k \\ &\quad + (-1)^n t + (-1)^{n+1} \end{aligned}$$

$$= (-1)^{n-1}(n-1)t + \sum_{k=2}^n (-1)^{n-k} \binom{n}{k} t^k$$

$$Y_1 = (-1)^{n-1}(n-1), Y_p = (-1)^{n-p} \binom{n}{p}, 2 \leq k \leq n.$$

By Lemma 2.1, then

$$\begin{aligned} \beta(C_n, 1) &= \sum_{p=1}^n Y_p S(p, 1) = Y_1 S(p, 1) + \sum_{p=2}^n Y_p S(p, 1) \\ &= (-1)^{n-1}(n-1) + \sum_{p=2}^n (-1)^{n-p} \binom{n}{p} = (-1)^n + (-1)^{n-1}n + \sum_{p=2}^n (-1)^{n-p} \binom{n}{p} \\ &= \sum_{p=0}^n (-1)^{n-p} \binom{n}{p} = (1-1)^n = 0, \end{aligned}$$

From Lemma 2.1,

$$\beta(C_n, k) = \sum_{p=k}^n (-1)^{n-p} \binom{n}{p} S(p, k),$$

$$2 \leq k \leq n.$$

**Lemma 2.4** ([1]) If  $G$  is the graph with  $n$  vertices, and the chromatic polynomial of  $G$  is  $f(G, t) = \sum_{p=1}^n Y_p t^p$ , then the number of all non-empty stable sets in  $G$  is as the following

$$\beta(G) = \sum_{p=1}^n Y_p b(p),$$

where  $b(p)$  is the  $p$ th Bell number.

**Lemma 2.5** If  $G$  is a cycle  $C_n$ , then

$$\beta(C_n) = \sum_{p=0}^n (-1)^{n-p} \binom{n}{p} b(p).$$

**Proof** According to the proving course of Lemma 2.3,  $Y_1 = (-1)^{n-1}(n-1)$ ,  $Y_p = (-1)^{n-p} \binom{n}{p}$ ,  $2 \leq k \leq n$ . By Lemma 2.4,

$$\begin{aligned} \beta(C_n) &= \sum_{p=1}^n Y_p b(p) = Y_1 b(1) + \sum_{p=2}^n Y_p b(p) \\ &= (-1)^{n-1}(n-1) + \sum_{p=2}^n (-1)^{n-p} \binom{n}{p} b(p) = (-1)^n b(0) + (-1)^{n-1} n b(1) + \\ &\quad + \sum_{p=2}^n (-1)^{n-p} \binom{n}{p} b(p) \\ &= \sum_{p=0}^n (-1)^{n-p} \binom{n}{p} b(p). \end{aligned}$$

**Lemma 2.6** ([1]) If  $G_1 \times G_2 \times \dots \times G_n$  is the direct product of graphs  $G_1, G_2, \dots, G_n$ , and  $\bar{G}_i \cap \bar{G}_j = \emptyset, i \neq j, 1 \leq i, j \leq n$ , then

$$\beta(G_1 \times G_2 \times \dots \times G_n, k) = \sum_{l_1+l_2+\dots+l_n=k} \beta(G_1, l_1)\beta(G_2, l_2)\dots\beta(G_n, l_n).$$

**Lemma 2.7** ([1]) If  $G_1 \times G_2 \times \dots \times G_n$  is the direct product of graphs  $G_1, G_2, \dots, G_n$ , and  $\bar{G}_i \cap \bar{G}_j = \emptyset, i \neq j, 1 \leq i, j \leq n$ , then

$$\beta(G_1 \times G_2 \times \dots \times G_n) = \beta(G_1)\beta(G_2)\dots\beta(G_n).$$

**Lemma 2.8** ([1]) There exists the relation between stable sets and  $S^{(n)}$ -factors

$$\beta(G, k) = N(\bar{G}, k).$$

**Lemma 2.9** ([1]) If  $A(\bar{G})$  is the number of all  $S^{(n)}$ -factors in  $\bar{G}$ , then

$$\beta(G) = A(\bar{G}).$$

**Lemma 2.10** ([2]) If  $P_n$  is a length  $n$  path with  $n + 1$  vertices, then

$$A(P_n) = F_{n+1},$$

where  $F_{n+1}$  is the  $(n + 1)$ th Fibonacci number.

Graphic interpretation: The  $n$ th Fibonacci number  $F_n$  is the number of all  $S^{(n)}$ -factors in  $P_{n-1}$  with  $n$  vertices.

**Lemma 2.11** ([2]) If  $C_n$  is a cycle with  $n$  vertices, then

$$A(C_n) = L_n,$$

$n \geq 4$ , where  $L_n$  is the  $n$ th Lucas number.

Graphic interpretation: Lucas number ( $n \geq 4$ ) is the number of all  $S^{(n)}$ -factors in  $C_n$ .

**Lemma 2.12** ([2]) If  $G$  is a graph with  $n$  vertices,  $P$  is a fixed vertex of the graph  $G$ , all complete graphs through the vertex  $P$  are  $K_{i_1}, K_{i_2}, \dots, K_{i_r}$ , then

$$N(G, k) = \sum_{j=1}^r N(G - V(K_{i_j}), k - 1),$$

where  $G - V(K_{i_j})$  is the graph that vertices  $V(K_{i_j})$  and these edges incident to  $V(K_{i_j})$  are all deleted.

**Lemma 2.13** ([2]) If  $G$  is a graph with  $n$  vertices,  $P$  is a fixed vertex of the graph  $G$ , all complete graphs through the vertex  $P$  are  $K_{i_1}, K_{i_2}, \dots, K_{i_r}$ , then

$$A(G) = \sum_{j=1}^r A(G - V(K_{i_j})),$$

where  $G - V(K_{i_j})$  is the graph that vertices  $V(K_{i_j})$  and these edges incident to  $V(K_{i_j})$  are all deleted.

**Lemma 2.14** ([3]) Suppose that  $P_n$  is a path with length  $n$ , and has  $n+1$  vertices, then

$$N(P_n, k) = \begin{cases} 0, & 1 \leq k < \frac{n+1}{2}, \\ \binom{k}{n+1-k}, & \frac{n+1}{2} \leq k \leq (n+1). \end{cases}$$

**Lemma 2.15** ([3]) Let  $C_n$  be a cycle with  $n$  vertices, and  $n \geq 4$ . Then

$$N(C_n, k) = \begin{cases} 0, & 1 \leq k < \frac{n}{2}, \\ \frac{n}{k} \binom{k}{n-k}, & \frac{n}{2} \leq k \leq n. \end{cases}$$

**Lemma 2.16** ([4]) If  $G$  is  $n-2$ -regular graph,  $n$  is even( $2m$ ), then

$$N(\bar{G}, k) = \begin{cases} 0, & 1 \leq k < \frac{n}{2}, \\ \binom{\frac{n}{2}}{k - \frac{n}{2}}, & \frac{n}{2} \leq k \leq n. \end{cases}$$

### 3. Main results

In the section, the author will research counting of partitions of stable sets of the direct product on several pairs of complementary graphs, and obtains their explicit formulas.

**Theorem 3.1** If  $G$  is the direct product of graphs  $P_{n_1}, P_{n_2}, \dots, P_{n_q}$ ,  $P_{n_i}$  is a length  $n_i$  path with  $n_i + 1$  vertices,  $1 \leq i \leq q$ , then

$$\beta(P_{n_1} \times P_{n_2} \times \dots \times P_{n_q}, k) = \sum_{l_1+l_2+\dots+l_q=k} \prod_{j=1}^q S(n_j, l_j - 1),$$

and

$$\beta(P_{n_1} \times P_{n_2} \times \dots \times P_{n_q}) = \prod_{j=1}^q b(n_j).$$

**Proof** For Lemma 2.6,

$$\beta(P_{n_1} \times P_{n_2} \times \dots \times P_{n_q}, k) = \sum_{l_1+l_2+\dots+l_n=k} \beta(P_{n_1}, l_1) \beta(P_{n_2}, l_2) \dots \beta(P_{n_q}, l_q).$$

Because path  $P_{n_i}$  is a special tree, by Lemma 2.2,  $\beta(P_{n_i}, l_i) = S(n_i, l_i - 1)$ ,  $1 \leq i \leq q$ .

Then

$$\begin{aligned}\beta(P_{n_1} \times P_{n_2} \times \dots \times P_{n_q}, k) &= \sum_{l_1+l_2+\dots+l_n=k} S(n_1, l_1 - 1)S(n_2, l_2 - 1)\dots S(n_q, l_q - 1) \\ &= \sum_{l_1+l_2+\dots+l_q=k} \prod_{j=1}^q S(n_j, l_j - 1).\end{aligned}$$

By Lemma 2.7,

$$\beta(P_{n_1} \times P_{n_2} \times \dots \times P_{n_q}) = \beta(P_{n_1})\beta(P_{n_2})\dots\beta(P_{n_q}).$$

From the proving course of above,  $\beta(P_{n_i}, l_i) = S(n_i, l_i - 1)$ ,  $1 \leq i \leq q$ .

$$\begin{aligned}\beta(P_{n_i}) &= \sum_{l_i=1}^{n_i+1} \beta(P_{n_i}, l_i) = \sum_{l_i=1}^{n_i+1} S(n_i, l_i - 1) \\ &= S(n_i, 0) + S(n_i, 1) + \dots + S(n_i, n_i) = b(n_i).\end{aligned}$$

Then

$$\begin{aligned}\beta(P_{n_1} \times P_{n_2} \times \dots \times P_{n_q}) &= b(n_1)b(n_2)\dots b(n_q) \\ &= \prod_{j=1}^q b(n_j).\end{aligned}$$

### Corollary 3.2

$$\beta(P_1 \times P_2 \times \dots \times P_n) = b(1)b(2)\dots b(n)$$

Graphic interpretation: the product of  $n$  continuous Bell numbers  $b(1)b(2)\dots b(n)$  is the number of all partitions of stable sets on the product graph  $P_1 \times P_2 \times \dots \times P_n$ .

**Theorem 3.3** If  $G$  is the direct product of complement paths  $\bar{P}_{n_1}, \bar{P}_{n_2}, \dots, \bar{P}_{n_q}$ , then

$$\beta(\bar{P}_{n_1} \times \bar{P}_{n_2} \times \dots \times \bar{P}_{n_q}, k) = \sum_{l_1+l_2+\dots+l_q=k} \prod_{j=1}^q \binom{l_j}{n_j + 1 - l_j},$$

and

$$\beta(\bar{P}_{n_1} \times \bar{P}_{n_2} \times \dots \times \bar{P}_{n_q}) = \prod_{j=1}^q F_{n_j+1}.$$

**Proof** By Lemma 2.6,

$$\beta(\bar{P}_{n_1} \times \bar{P}_{n_2} \times \dots \times \bar{P}_{n_q}, k) = \sum_{l_1+l_2+\dots+l_n=k} \beta(\bar{P}_{n_1}, l_1)\beta(\bar{P}_{n_2}, l_2)\dots\beta(\bar{P}_{n_q}, l_q).$$

By Lemma 2.8,

$$\beta(\bar{P}_{n_j}, l_j) = N(P_{n_j}, l_j).$$

For Lemma 2.14,

$$N(P_{n_j}, l_j) = \binom{l_j}{n_j + 1 - l_j},$$

hence

$$\beta(\bar{P}_{n_j}, l_j) = \binom{l_j}{n_j + 1 - l_j}.$$

Then

$$\begin{aligned} & \beta(\bar{P}_{n_1} \times \bar{P}_{n_2} \times \dots \times \bar{P}_{n_q}, k) \\ = & \sum_{l_1 + l_2 + \dots + l_q = k} \binom{l_1}{n_1 + 1 - l_1} \binom{l_2}{n_2 + 1 - l_2} \dots \binom{l_q}{n_q + 1 - l_q} \\ = & \sum_{l_1 + l_2 + \dots + l_q = k} \prod_{j=1}^q \binom{l_j}{n_j + 1 - l_j}. \end{aligned}$$

With Lemma 2.7,

$$\beta(\bar{P}_{n_1} \times \bar{P}_{n_2} \times \dots \times \bar{P}_{n_q}) = \beta(\bar{P}_{n_1})\beta(\bar{P}_{n_2})\dots\beta(\bar{P}_{n_q}).$$

By Lemma 2.9,

$$\beta(\bar{P}_{n_j}) = A(P_{n_j}).$$

For Lemma 2.10,

$$A(P_{n_j}) = F_{n_j+1}.$$

Then

$$\beta(\bar{P}_{n_1} \times \bar{P}_{n_2} \times \dots \times \bar{P}_{n_q}) = \prod_{j=1}^q F_{n_j+1}.$$

### Corollary 3.4

$$\beta(\bar{P}_0 \times \bar{P}_1 \times \dots \times \bar{P}_{n-1}) = F_1 F_2 \dots F_n.$$

**Proof** According to Theorem 3.3, let  $n_1 = 0, n_2 = 1, \dots, n_q = n - 1$ , so,

$$\beta(\bar{P}_0 \times \bar{P}_1 \times \dots \times \bar{P}_{n-1}) = F_1 F_2 \dots F_n.$$

Graphic interpretation: the product of  $n$  continuous Fibonacci numbers  $F_1 F_2 \dots F_n$  is the number of all partitions of stable sets on the product graph  $\bar{P}_0 \times \bar{P}_1 \times \dots \times \bar{P}_{n-1}$ .

**Theorem 3.5** If  $G$  is the direct product of cycles  $C_{n_1}, C_{n_2}, \dots, C_{n_q}$ ,  $C_{n_i}$  is a cycle with  $n_i$  vertices,  $1 \leq i \leq q$ , then

$$\beta(C_{n_1} \times C_{n_2} \times \dots \times C_{n_q}, k) = \sum_{\sum_{j=1}^q l_j = k} \prod_{j=1}^q \sum_{p=l_j}^{n_j} (-1)^{n_j-p} \binom{n_j}{p} S(p, l_j),$$

and

$$\beta(C_{n_1} \times C_{n_2} \times \dots \times C_{n_q}) = \prod_{j=1}^q \sum_{p=0}^{n_j} (-1)^{n_j-p} \binom{n_j}{p} b(p).$$

**Proof** By Lemma 2.6,

$$\beta(C_{n_1} \times C_{n_2} \times \dots \times C_{n_q}, k) = \sum_{l_1 + l_2 + \dots + l_n = k} \beta(C_{n_1}, l_1) \beta(C_{n_2}, l_2) \dots \beta(C_{n_q}, l_q).$$

For Lemma 2.3,

$$\beta(C_{n_j}, 1) = 0$$

$$\beta(C_{n_j}, k) = \sum_{p=k}^{n_j} (-1)^{n_j-p} \binom{n_j}{p} S(p, k),$$

$2 \leq k \leq n$ , then

$$\begin{aligned} & \beta(C_{n_1} \times C_{n_2} \times \dots \times C_{n_q}, k) \\ = & \sum_{\substack{q \\ j=1}} \sum_{l_j=k, 2 \leq l_j \leq k} \sum_{p=l_1}^{n_1} (-1)^{n_1-p} \binom{n_1}{p} S(p, l_1) \sum_{p=l_2}^{n_2} (-1)^{n_2-p} \binom{n_2}{p} S(p, l_2) \dots \\ & \sum_{p=l_q}^{n_q} (-1)^{n_q-p} \binom{n_q}{p} S(p, l_q) \\ = & \sum_{\substack{q \\ j=1}} \sum_{l_j=k, 2 \leq l_j \leq k} \prod_{j=1}^q \sum_{p=l_j}^{n_j} (-1)^{n_j-p} \binom{n_j}{p} S(p, l_j). \end{aligned}$$

By Lemma 2.7,

$$\beta(C_{n_1} \times C_{n_2} \times \dots \times C_{n_q}) = \beta(C_{n_1})\beta(C_{n_2})\dots\beta(C_{n_q}).$$

For Lemma 2.5,

$$\beta(C_{n_j}) = \sum_{p=0}^{n_j} (-1)^{n_j-p} \binom{n_j}{p} b(p),$$

$1 \leq l_j \leq q$

Then

$$\begin{aligned} & \beta(C_{n_1} \times C_{n_2} \times \dots \times C_{n_q}) \\ = & \sum_{p=0}^{n_1} (-1)^{n_1-p} \binom{n_1}{p} b(p) \sum_{p=0}^{n_2} (-1)^{n_2-p} \binom{n_2}{p} b(p) \dots \sum_{p=0}^{n_q} (-1)^{n_q-p} \binom{n_q}{p} b(p) \\ = & \prod_{j=1}^q \sum_{p=0}^{n_j} (-1)^{n_j-p} \binom{n_j}{p} b(p). \end{aligned}$$

**Theorem 3.6** If  $G$  is the direct product of complement cycles  $\bar{C}_{n_1}, \bar{C}_{n_2}, \dots, \bar{C}_{n_q}$ , then

$$\beta(\bar{C}_{n_1} \times \bar{C}_{n_2} \times \dots \times \bar{C}_{n_q}, k) = \sum_{l_1+l_2+\dots+l_q=k} \prod_{j=1}^q N(C_{n_j}, l_j),$$

$$\text{when } n_j = 3, N(C_3, l) = \begin{cases} 1, & l = 1, \\ 3, & l = 2, \\ 1, & l = 3, \end{cases}$$

$$\text{when } n_j \geq 4, N(C_{n_j}, l_j) = \begin{cases} 0, & 1 \leq l_j < \frac{n_j}{2}, \\ \frac{n_j}{l_j} \binom{l_j}{n_j - l_j}, & \frac{n_j}{2} \leq l_j \leq n_j, \end{cases}$$

and

$$\beta(\bar{C}_{n_1} \times \bar{C}_{n_2} \times \dots \times \bar{C}_{n_q}) = 5^l \prod_{j=1}^{q-l} L_{n_j},$$

where the number of  $l_j = 3$  is 1,  $1 \leq j \leq q$ ,  $L_{n_j}$  is the  $n_j$ th Lucas number.

**Proof** According to Lemma 2.6,

$$\beta(\bar{C}_{n_1} \times \bar{C}_{n_2} \times \dots \times \bar{C}_{n_q}, k) = \sum_{l_1+l_2+\dots+l_n=k} \beta(\bar{C}_{n_1}, l_1) \beta(\bar{C}_{n_2}, l_2) \dots \beta(\bar{C}_{n_q}, l_q).$$

By Lemma 2.8,

$$\beta(\bar{C}_{n_j}, l_j) = N(C_{n_j}, l_j).$$

Then

$$\beta(\bar{C}_{n_1} \times \bar{C}_{n_2} \times \dots \times \bar{C}_{n_q}, k) = \sum_{l_1+l_2+\dots+l_n=k} N(C_{n_1}, l_1) N(C_{n_2}, l_2) \dots N(C_{n_q}, l_q),$$

where

$$\text{when } n_j = 3, N(C_3, l) = \begin{cases} 1, & l = 1, \\ 3, & l = 2, \\ 1, & l = 3, \end{cases}$$

$$\text{when } n_j \geq 4, N(C_{n_j}, l_j) = \begin{cases} 0, & 1 \leq l_j < \frac{n_j}{2}, \\ \frac{n_j}{l_j} \binom{l_j}{n_j - l_j}, & \frac{n_j}{2} \leq l_j \leq n_j, \end{cases}$$

see Lemma 2.15.

By Lemma 2.7,

$$\beta(\bar{C}_{n_1} \times \bar{C}_{n_2} \times \dots \times \bar{C}_{n_q}) = \beta(\bar{C}_{n_1}) \beta(\bar{C}_{n_2}) \dots \beta(\bar{C}_{n_q}).$$

Because of  $\beta(\bar{C}_3) = A(K_3) = 5$ , by Lemma 2.9 and Lemma 2.11,  $\beta(\bar{C}_{n_j}) = A(C_{n_j}) = L_{n_j}$ , and the number of  $l_j = 3$  is 1,  $1 \leq j \leq q$ , then

$$\begin{aligned} \beta(\bar{C}_{n_1} \times \bar{C}_{n_2} \times \dots \times \bar{C}_{n_q}) &= \beta(\bar{C}_3)^l \prod_{j=1}^{q-l} \beta(\bar{C}_{n_j}) \\ &= 5^l \prod_{j=1}^{q-l} L_{n_j}. \end{aligned}$$

**Theorem 3.7** If  $G$  is the direct product of graphs  $n_1 - 2$ -regular graph,  $n_2 - 2$ -regular graph, ...,  $n_q - 2$ -regular graph, and  $n_j$  is even( $2m_j$ ),  $1 \leq j \leq q$ , then

$$\beta(G, k) = \sum_{l_1+l_2+\dots+l_q=k} \binom{m_1}{l_1-m_1} \binom{m_2}{l_2-m_2} \dots \binom{m_q}{l_q-m_q},$$

and

$$\beta(G) = 2^{m_1+m_2+\dots+m_q}.$$

**Proof** Let  $G = G_1 \times G_2 \times \dots \times G_q$ ,  $G_j$  is  $n_j - 2$ -regular graph, and  $n_j$  is even( $2m_j$ ),  $1 \leq j \leq q$ . With Lemma 2.6 and Lemma 2.8,

$$\begin{aligned} \beta(G_1 \times G_2 \times \dots \times G_q, k) &= \sum_{l_1+l_2+\dots+l_n=k} \beta(G_1, l_1) \beta(G_2, l_2) \dots \beta(G_q, l_q) \\ &= \sum_{l_1+l_2+\dots+l_n=k} N(\bar{G}_1, l_1) N(\bar{G}_2, l_2) \dots N(\bar{G}_q, l_q). \end{aligned}$$

Because  $G_j$  is  $n_j - 2$ -regular graph,  $\bar{G}_j$  is 1-regular graph. Thus assume that

$$\bar{G}_j = K_2 \cup K_2 \cup \dots \cup K_2$$

say. Here the number of  $K_2$  is  $m_j$ . For Lemma 2.16,  $N(\bar{G}_j, l_j) = \binom{m_j}{l_j-m_j}$ ,  $1 \leq j \leq q$ , then

$$\beta(G, k) = \sum_{l_1+l_2+\dots+l_q=k} \binom{m_1}{l_1-m_1} \binom{m_2}{l_2-m_2} \dots \binom{m_q}{l_q-m_q}.$$

By Lemma 2.7,

$$\beta(G) = \beta(G_1 \times G_2 \times \dots \times G_q) = \beta(G_1) \beta(G_2) \dots \beta(G_q).$$

By Lemma 2.8,

$$\begin{aligned} \beta(G_j) &= \sum_{j=1}^{n_j} \beta(G_j, j) = \sum_{j=1}^{n_j} N(\bar{G}_j, j) = \sum_{j=1}^{n_j} \binom{m_j}{j-m_j} = \sum_{j=m_j}^{2(m_j)} \binom{m_j}{j-m_j} \\ &= \sum_{j=0}^{m_j} \binom{m_j}{p} = 2^{m_j}. \end{aligned}$$

Then

$$\beta(G) = 2^{m_1} 2^{m_2} \dots 2^{m_q} = 2^{m_1+m_2+\dots+m_q}.$$

**Theorem 3.8**  $G$  is the direct product of  $n_j - 2$ -regular complement graphs,  $1 \leq j \leq q$ ,  $n_j$  is even( $2m_j$ ), then

$$\beta(G, k) = \sum_{l_1+l_2+\dots+l_q=k} \prod_{j=1}^q \sum_{p=l_j}^{2m_j} (-1)^{2m_j-p} \binom{m_j}{2m_j-p} S(p, l_j),$$

and

$$\beta(G) = \prod_{j=1}^q \sum_{p=1}^{2m_j} (-1)^{2m_j-p} \binom{m_j}{2m_j-p} b(p).$$

**Proof** Let  $G_j$  be  $n_j - 2$ -regular graph,  $\bar{G}_j$  be  $n_j - 2$ -regular complement graphs. Then  $\bar{G}_j$  is 1-regular graph. Thus assume that  $\bar{G}_j = K_2 \cup K_2 \cup \dots \cup K_2$  say. The number of  $K_2$  graphs is  $m_j$ . By Lemma 2.6,

$$\beta(\bar{G}_1 \times \bar{G}_2 \times \dots \times \bar{G}_q, k) = \sum_{l_1+l_2+\dots+l_q=k} \beta(\bar{G}_1, l_1) \beta(\bar{G}_2, l_2) \dots \beta(\bar{G}_q, l_q).$$

The chromatic polynomial

$$\begin{aligned} f(\bar{G}_j, t) &= f(K_2, t) f(K_2, t) \dots f(K_2, t) \\ &= t(t-1)t(t-1) \dots t(t-1) \\ &= t^{m_j} (t-1)^{m_j} \\ &= t^{m_j} \sum_{j=0}^{m_j} (-1)^{m_j-j} \binom{m_j}{j} t^j \\ &= \sum_{j=0}^{m_j} (-1)^{m_j-j} \binom{m_j}{j} t^{m_j+j}, \end{aligned}$$

let  $m_j + j = p$ ,

$$\begin{aligned} f(\bar{G}_j, t) &= \sum_{p=m_j}^{2m_j} (-1)^{2m_j-p} \binom{m_j}{p-m_j} t^p \\ &= \sum_{p=m_j}^{2m_j} (-1)^{2m_j-p} \binom{m_j}{2m_j-p} t^p, \end{aligned}$$

when  $1 \leq p \leq m_j$ ,  $Y_p = 0$ ; when  $m_j \leq p \leq 2m_j$ ,  $Y_p = (-1)^{2m_j-p} \binom{m_j}{2m_j-p}$ .

By Lemma 2.1,

$$\beta(\bar{G}_j, k) = \sum_{p=k}^{2m_j} (-1)^{2m_j-p} \binom{m_j}{2m_j-p} S(p, k).$$

Then

$$\begin{aligned} \beta(G, k) &= \beta(\bar{G}_1 \times \bar{G}_2 \times \dots \times \bar{G}_q, k) = \sum_{l_1+l_2+\dots+l_q=k} \sum_{p=l_1}^{2m_j} (-1)^{2m_j-p} \binom{m_j}{2m_j-p} S(p, l_1) \sum_{p=l_2}^{2m_j} (-1)^{2m_j-p} \binom{m_j}{2m_j-p} \\ &\quad \sum_{p=l_q}^{2m_j} (-1)^{2m_j-p} \binom{m_j}{2m_j-p} S(p, l_q) \\ &= \sum_{l_1+l_2+\dots+l_q=k} \prod_{j=1}^q \sum_{p=l_j}^{2m_j} (-1)^{2m_j-p} \binom{m_j}{2m_j-p} S(p, l_j). \end{aligned}$$

By Lemma 2.7,

$$\beta(\bar{G}_1 \times \bar{G}_2 \times \dots \times \bar{G}_q) = \beta(\bar{G}_1)\beta(\bar{G}_2)\dots\beta(\bar{G}_q).$$

From the proving course of above, according to Lemma 2.4,

$$\beta(\bar{G}_j) = \sum_{p=1}^{2m_j} (-1)^{2m_j-p} \binom{m_j}{2m_j-p} b(p).$$

Then

$$\begin{aligned} \beta(G) &= \beta(\bar{G}_1 \times \bar{G}_2 \times \dots \times \bar{G}_q) \\ &= \sum_{p=1}^{2m_1} (-1)^{2m_1-p} \binom{m_1}{2m_1-p} b(p) \sum_{p=1}^{2m_2=2} (-1)^{2m_2-p} \binom{m_2}{2m_2-p} b(p) \dots \\ &\quad \sum_{p=1}^{2m_q} (-1)^{2m_q-p} \binom{m_q}{2m_q-p} b(p) \\ &= \prod_{j=1}^q \sum_{p=1}^{2m_j} (-1)^{2m_j-p} \binom{m_j}{2m_j-p} b(p). \end{aligned}$$

**Theorem 3.9** If  $G$  is a wheel graph  $O \odot C_n$ , then  $\beta(O \odot C_n, 1) = 0$ ,  $\beta(O \odot C_n, 2) = 0$ ,

$$\beta(O \odot C_n, k) = \sum_{p=k-1}^n (-1)^{n-p} \binom{n}{p} S(p, k-1),$$

$3 \leq k \leq n+1$ , and

$$\beta(O \odot C_n) = \sum_{p=0}^n (-1)^{n-p} \binom{n}{p} b(p).$$

**Proof** For

$$\beta(O \odot C_n, k) = \beta(O \times C_n, k),$$

by Lemma 2.6,

$$\beta(O \odot C_n, k) = \sum_{l+m=k} \beta(O, l) \beta(C_n, m) = \beta(C_n, k-1).$$

Because of  $O \odot C_n \neq O_{n+1}$ ,  $\beta(O \odot C_n, 1) = 0$ .

With Lemma 2.3,

$$\begin{aligned} \beta(C_n, 1) &= 0 \\ \beta(C_n, k) &= \sum_{p=k}^n (-1)^{n-p} \binom{n}{p} S(p, k), \end{aligned}$$

$2 \leq k \leq n$ .

$$\begin{aligned} \beta(O \odot C_n, 2) &= \beta(C_n, 2-1) = 0, \\ \beta(C_n, k-1) &= \sum_{p=k-1}^n (-1)^{n-p} \binom{n}{p} S(p, k-1), \end{aligned}$$

$3 \leq k \leq n$ .

Then

$$\beta(O \odot C_n, k) = \sum_{p=k-1}^n (-1)^{n-p} \binom{n}{p} S(p, k-1),$$

$3 \leq k \leq n+1$ .

By Lemma 2.7,

$$\beta(O \odot C_n) = \beta(O)\beta(C_n) = \beta(C_n).$$

For Lemma 2.5,

$$\beta(C_n) = \sum_{p=0}^n (-1)^{n-p} \binom{n}{p} b(p).$$

Then

$$\beta(O \odot C_n) = \sum_{p=0}^n (-1)^{n-p} \binom{n}{p} b(p).$$

**Theorem 3.10** If  $G$  is the complement graph of a wheel graph  $O \odot C_n$ , then

$$\beta(O \bar{\odot} C_n, k) = \frac{n}{k-1} \binom{k-1}{n-k+1} + n \binom{k}{n-k},$$

and when  $n = 3$ ,  $\beta(O \bar{\odot} C_n) = 14$ ;

when  $n \geq 4$ ,

$$\beta(O \bar{\odot} C_n) = L_n + nF_n.$$

**Proof** Conduct analyzing for vertex  $O$ , all complete graphs through vertex  $O$  are  $K_1; K_2, K_2, \dots, K_2; K_3, K_3, \dots, K_3$ . The number of  $K_2$  graphs is  $n$ , and the number of  $K_3$  graphs is also  $n$ .

Case 1,  $O$  as  $K_1$ , delete vertex  $K_1$ , then there is a cycle  $C_n$ ,

$$N(G - V(K_1), k-1) = N(C_n, k-1).$$

Case 2,  $O$  is contained in  $K_2$ , delete  $K_2$ , then there is a path  $P_{n-2}$ ,

$$N(G - V(K_2), k-1) = N(P_{n-2}, k-1).$$

Because the number of  $K_2$  is  $n$ , and these  $K_2$  graphs are symmetric,

$$nN(G - V(K_2), k-1) = nN(P_{n-2}, k-1).$$

Case 3,  $O$  is contained in  $K_3$ , delete  $K_3$ , then there is a path  $P_{n-3}$ ,

$$N(G - V(K_3), k-1) = N(P_{n-3}, k-1).$$

Because the number of  $K_3$  is  $n$ , and these  $K_3$  graphs are symmetric,

$$nN(G - V(K_3), k-1) = nN(P_{n-3}, k-1).$$

Summarize above, according to Lemma 2.12,

$$\beta(O \bar{\odot} C_n, k) = N(O \odot C_n, k) = \sum_{j=1}^r N(G - V(K_{i_j}), k-1)$$

$$\begin{aligned}
 &= N(G - V(K_1), k - 1) + nN(G - V(K_2), k - 1) + nN(G - V(K_3), k - 1) \\
 &= N(C_n, k - 1) + nN(P_{n-2}, k - 1) + nN(P_{n-3}, k - 1).
 \end{aligned}$$

By Lemma 2.14 and Lemma 2.15,

$$\begin{aligned}
 \beta(O \bar{\odot} C_n, k) &= \frac{n}{k-1} \binom{k-1}{n-k+1} + n \binom{k-1}{n-k} + n \binom{k-1}{n-k-1} \\
 &= \frac{n}{k-1} \binom{k-1}{n-k+1} + n \binom{k}{n-k}.
 \end{aligned}$$

By Lemma 2.9,

$$\beta(O \bar{\odot} C_n) = A(O \odot C_n).$$

Analogous to above, according to Lemma 2.13,

$$\begin{aligned}
 A(O \odot C_n) &= \sum_{j=1}^r N(G - V(K_{i_j})) \\
 &= A(G - V(K_1)) + nA(G - V(K_2)) + nA(G - V(K_3)) \\
 &= A(C_n) + nA(P_{n-2}) + nA(P_{n-3}),
 \end{aligned}$$

when  $n = 3$ ,

$$\beta(O \bar{\odot} C_3) = A(C_3) + 3A(P_1) + 3A(P_0) = 5 + 3 \times 2 + 3 \times 1 = 14;$$

when  $n \geq 4$ , by Lemma 2.10 and Lemma 2.11,

$$\beta(O \bar{\odot} C_n) = L_n + nF_{n-1} + nF_{n-2} = L_n + n(F_{n-1} + F_{n-2}) = L_n + nF_n.$$

**Remark 3.11** If  $G \cap H = \phi$ , then

$$\beta(G \cup H, k) \neq \sum_{l+m=k} \beta(G, l)\beta(H, m),$$

and

$$\beta(G \cup H) \neq \beta(G)\beta(H).$$

**Proof** For

$$\beta(\bar{O} \cup \bar{C}_n, k) = \beta(O \bar{\odot} C_n, k),$$

by Theorem 3.10,

$$\beta(\bar{O} \cup \bar{C}_n, k) = \frac{n}{k-1} \binom{k-1}{n-k+1} + n \binom{k}{n-k},$$

and

$$\sum_{l+m=k} \beta(\bar{O}, l)\beta(\bar{C}_n, m) = \beta(\bar{C}_n, k-1),$$

by Lemma 2.8 and Lemma 2.15,

$$\sum_{l+m=k} \beta(\bar{O}, l)\beta(\bar{C}_n, m) = N(C_n, k-1) = \frac{n}{k-1} \binom{k-1}{n-k+1},$$

so,

$$\beta(\bar{O} \cup \bar{C}_n, k) \neq \sum_{l+m=k} \beta(\bar{O}, l)\beta(\bar{C}_n, m).$$

With

$$\beta(\bar{O} \cup \bar{C}_n) = \beta(O \odot C_n),$$

by Theorem 3.10,

$$\beta(\bar{O} \cup \bar{C}_n) = L_n + nF_n,$$

and

$$\beta(\bar{O})\beta(\bar{C}_n) = \beta(\bar{C}_n),$$

by Lemma 2.9 and Lemma 2.11,

$$\beta(\bar{O})\beta(\bar{C}_n) = A(C_n) = L_n,$$

so,

$$\beta(\bar{O} \cup \bar{C}_n) \neq \beta(\bar{O})\beta(\bar{C}_n).$$

#### 4. Conclutions

In this paper, the authors further study the counting of stable sets, and obtains explicit formulas of partitions of stable sets of the direct product on several pairs of complementary graphs, this study is of reference value to combination counting and graph theory counting.

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