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A UNIMODAL CONJECTURE RELATED THE QUADRUPLE SEQUENCE ON THE SERIES

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ABSTRACT. The content of this article is the application of enumeration combinatorics in economics. Xiaoping Deng, the great leader and chief designer of China, proposed that China's fiscal revenue should quadruple. The authors are inspired to obtain the quadruple sequence, further researches its generating functions within the framework of the mathematical model, and give a unimodal conjecture related the quadruple sequence about the series, also it has been proven that this conjecture holds when $1 \leq r \leq 11$. The research has both theoretical value and practical significance for combinatorics, mathematical analysis and economics.

1. Introduction

In [1], 1968, mathematician R C. Read proposed the hypothesis that the sequence of absolute values of the coefficients of the chromatic polynomial forms unimodal. This conjecture has been proven by June Huh, a professor at Princeton University who won the Fields Prize in Mathematics in 2022, using the Hodge Theory Matroid method in algebraic geometry. His results have been published in the top ten mathematical journals in the world, as shown in [2]. In [3], G Boros and V H. Moll proved that if the coefficients of a polynomial $P(x)$ are non negative and non decreasing, then $P(x + 1)$ is unimodal.

Deng Xiaoping proposed in 1984 that China's industrial and agricultural value should quadruple by 2000 compared to 1987, which is $2^2 = 4$ times the original value. From this, we are inspired to discover a special sequence, quadruple, then quadruple again, ..., the n th quadruple of a sequence, i.e.

Namely $1, 4, 4^2, 4^3, \dots, 4^n, \dots$

or $1, 4, 16, 64, 256, 1024, \dots, 4^n, \dots$,

this sequence is called a quadruple sequence.

The general term of the quadruple sequence:

$$a_n = 4^n, n = 0, 1, 2, 3, \dots, n, \dots$$

The generating function of the quadruple sequence:

$$f(x) = \sum_{n=0}^{\infty} 4^n x^n = \sum_{n=0}^{\infty} (4x)^n = \frac{1}{1-4x}, |x| < \frac{1}{4}$$

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The sum of the first n terms in the quadruple sequence:

$$S_n = 1 + 4 + 4^2 + 4^3 + \dots + 4^{n-1} = \frac{1 - 4^n}{1 - 4} = \frac{1}{3}(4^n - 1)$$

Overall analysis of the quadruple of national fiscal revenue:

1 Quadruple for the first time ($2^2 = 4$ times)

In 2000, the total national fiscal revenue was 1.339523 trillion yuan, breaking through the billion yuan mark, in 2006, China's total fiscal revenue was 3.93732 trillion yuan, achieved the first quadrupling of 4.0 trillion yuan.

2 Quadruple for the second time ($4^2 = 16$ times)

From 2006 to 2022, the total national fiscal revenue reached 10.5687 trillion yuan, which was still some distance away from the second quadruple of 16.00 trillion yuan. It is expected to achieve a second quadruple by 2025. In fact, the total fiscal revenue in 2023 was 21.6784 trillion yuan, which has exceeded the second doubling.

3 Quadruple for the third time ($4^3 = 64$ times)

The third time we will quadruple, the great goal is 64.00 trillion yuan. In a few years, China can also achieve this third time we will quadruple.

4 Quadruple for the fourth time ($4^4 = 256$ times)

Realizing a national total fiscal revenue of 256.00 trillion yuan is not impossible, but hopeful.

5 Quadruple for the fifth time ($4^5 = 1024$ times)

To achieve the fifth quadruple of 1024.00 trillion yuan, this great goal will make China the world's economic powerhouse.

If developed according to this economic model, China will become the world's first economic powerhouse in the future .

The mathematical problems arising from this are as the following.

Conjecture

1 When r is odd, $r, m \in N_+$,

$$\sum_{n=0}^{\infty} \frac{n^r}{2^{mn}} = \frac{1}{(2^m - 1)^{r+1}} (2^m + a_2 \cdot 2^{2m} + a_3 \cdot 2^{3m} + \dots +$$

$$+ a_{[\frac{r}{2}]} \cdot 2^{[\frac{r}{2}]m} + a_{[\frac{r}{2}]+1} \cdot 2^{([\frac{r}{2}]+1)m} + a_{[\frac{r}{2}]} \cdot 2^{([\frac{r}{2}]+2)m} + \dots + a_2 \cdot 2^{(r-1)m} + 2^{rm}),$$

where (1) Base: $2^m, 2^{2m}, \dots, 2^{(r-1)m}, 2^{rm}$

(2) Sequence of its coefficients: $1, a_2, a_3, \dots, a_{[\frac{r}{2}]}, a_{[\frac{r}{2}]+1}, a_{[\frac{r}{2}]}, \dots, a_2, 1$, multiply separately $\frac{1}{(2^m - 1)^{r+1}}$

or $a_1, a_2, a_3, \dots, a_{r-1}, a_r$ multiply separately $\frac{1}{(2^m - 1)^{r+1}}$

conjecture: the sequence is unimodal and symmetric($a_k = a_{r+1-k}, 1 \leq k \leq r$)?

(3) Peak: only one, $\frac{1}{(2^m - 1)^{r+1}} a_{[\frac{r}{2}]+1}$.

2 When r is even, $r, m \in N_+$,

$$\sum_{n=0}^{\infty} \frac{n^r}{2^{mn}} = \frac{1}{(2^m - 1)^{r+1}} (2^m + a_2 \cdot 2^{2m} + a_3 \cdot 2^{3m} + \dots + a_{[\frac{r}{2}]-1} \cdot 2^{([\frac{r}{2}]-1)m} +$$

$$+ a_{[\frac{r}{2}]} \cdot 2^{[\frac{r}{2}]m} + a_{[\frac{r}{2}]+1} \cdot 2^{([\frac{r}{2}]+1)m} + a_{[\frac{r}{2}]-1} \cdot 2^{([\frac{r}{2}]+2)m} + \dots + a_2 \cdot 2^{(r-1)m} + 2^{rm}),$$

where (1) Base: $2^m, 2^{2m}, \dots, 2^{(r-1)m}, 2^{rm}$

(2) Sequence of its coefficients:

$1, a_2, a_3, \dots, a_{[\frac{r}{2}]-1}, a_{[\frac{r}{2}]}, a_{[\frac{r}{2}]+1}, a_{[\frac{r}{2}]-1}, \dots, a_2, 1,$

multiply separately $\frac{1}{(2^m-1)^{r+1}}$

or $a_1, a_2, a_3, \dots, a_{r-1}, a_r$ multiply separately $\frac{1}{(2^m-1)^{r+1}}$

conjecture: the sequence is unimodal and symmetric ($a_k = a_{r+1-k}, 1 \leq k \leq r$)?

(3) Peak: with two peaks, $\frac{1}{(2^m-1)^{r+1}} a_{[\frac{r}{2}]}$ and $\frac{1}{(2^m-1)^{r+1}} a_{[\frac{r}{2}]+1},$

also $a_{[\frac{r}{2}]} = a_{[\frac{r}{2}]+1}.$

(also see [4 - 5 - 6 - 7 - 8])

This conjecture is an interesting rule, similar to the famous Nemton binomial theorem, where the coefficients are unimodal and symmetric, and the peak points are one or two.

Some results on the quadruple sequence are described as follows.

Theorem 1.1. *The generating function of the sequence $\{4^n \cdot n\}$ is*

$$f(x) = \sum_{n=0}^{\infty} 4^n \cdot n x^n = \frac{4x}{(1-4x)^2}, |x| < \frac{1}{4}.$$

Theorem 1.2. *There exists the sum equation*

$$\frac{1}{2^m} + \frac{2}{2^{2m}} + \dots + \frac{n}{2^{mn}} + \dots = \frac{2^m}{(2^m-1)^2}, m \in N_+.$$

Theorem 1.3. *The generating function of the sequence $\{4^n \cdot n^2\}$ is*

$$f(x) = \sum_{n=0}^{\infty} 4^n \cdot n^2 x^n = \frac{x(16x+4)}{(1-4x)^3}, |x| < \frac{1}{4}.$$

Theorem 1.4. *There is the sum equation*

$$\frac{1^2}{2^m} + \frac{2^2}{2^{2m}} + \dots + \frac{n^2}{2^{mn}} + \dots = \frac{2^m + 2^{2m}}{(2^m-1)^3}, m \in N_+.$$

Theorem 1.5. *The generating function of the sequence $\{4^n \cdot n^3\}$ is*

$$f(x) = \sum_{n=0}^{\infty} 4^n \cdot n^3 x^n = \frac{x(64x^2 + 64x + 4)}{(1-4x)^4}, |x| < \frac{1}{4}.$$

Theorem 1.6. *There is the sum identity*

$$\frac{1^3}{2^m} + \frac{2^3}{2^{2m}} + \dots + \frac{n^3}{2^{mn}} + \dots = \frac{2^m + 4 \cdot 2^{2m} + 2^{3m}}{(2^m-1)^4}, m \in N_+.$$

Theorem 1.7. *The generating function of the sequence $\{4^n \cdot n^4\}$ is*

$$f(x) = \sum_{n=0}^{\infty} 4^n \cdot n^4 x^n = \frac{x(256x^3 + 704x^2 + 176x + 4)}{(1-4x)^5}, |x| < \frac{1}{4}.$$

Theorem 1.8. *There exists the sum identity*

$$\frac{1^4}{2^m} + \frac{2^4}{2^{2m}} + \dots + \frac{n^4}{2^{mn}} + \dots = \frac{2^m + 11 \cdot 2^{2m} + 11 \cdot 2^{3m} + 2^{4m}}{(2^m-1)^5}, m \in N_+.$$

Theorem 1.9. *The generating function of the sequence $\{4^n \cdot n^5\}$ is*

$$f(x) = \sum_{n=0}^{\infty} 4^n \cdot n^5 x^n = \frac{x(1024x^4 + 6656x^3 + 4224x^2 + 416x + 4)}{(1-4x)^6}, |x| < \frac{1}{4}.$$

Theorem 1.10. *There exists the sum identity*

$$\frac{1^5}{2^m} + \frac{2^5}{2^{2m}} + \cdots + \frac{n^5}{2^{mn}} + \cdots = \frac{2^m + 26 \cdot 2^{2m} + 66 \cdot 2^{3m} + 26 \cdot 2^{4m} + 2^{5m}}{(2^m - 1)^6}, m \in N_+.$$

Theorem 1.11. *The generating function of the sequence $\{4^n \cdot n^6\}$ is*

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} 4^n \cdot n^6 x^n = \\ &= \frac{x}{(1-4x)^7} (4096x^5 + 58368x^4 + 77312x^3 + 19328x^2 + 912x + 4), |x| < \frac{1}{4}. \end{aligned}$$

Theorem 1.12. *There exists the sum identity*

$$\begin{aligned} \frac{1^6}{2^m} + \frac{2^6}{2^{2m}} + \cdots + \frac{n^6}{2^{mn}} + \cdots = \\ = \frac{2^m + 57 \cdot 2^{2m} + 302 \cdot 2^{3m} + 302 \cdot 2^{4m} + 57 \cdot 2^{5m} + 2^{6m}}{(2^m - 1)^7}, m \in N_+. \end{aligned}$$

Theorem 1.13. *The generating function of series $\{4^n \cdot n^7\}$ is*

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} 4^n \cdot n^7 x^n = \\ &= \frac{x}{(1-4x)^8} (16384x^6 + 491520x^5 + 1219584x^4 + 618496x^3 + 76224x^2 + 1920x + 4), \\ &|x| < \frac{1}{4}. \end{aligned}$$

Theorem 1.14. *There exists the sum identity*

$$\begin{aligned} \frac{1^7}{2^m} + \frac{2^7}{2^{2m}} + \cdots + \frac{n^7}{2^{mn}} + \cdots = \\ = \frac{2^m + 120 \cdot 2^{2m} + 1191 \cdot 2^{3m} + 2416 \cdot 2^{4m} + 1191 \cdot 2^{5m} + 120 \cdot 2^{6m} + 2^{7m}}{(2^m - 1)^8}, \end{aligned}$$

$m \in N_+$.

Theorem 1.15. *The generating function of series $\{4^n \cdot n^8\}$ is*

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} 4^n \cdot n^8 x^n = \\ &= \frac{x}{(1-4x)^9} (65536x^7 + 4046848x^6 + 17584128x^5 + 15993856x^4 + 3998464x^3 + \\ &\quad + 274752x^2 + 3952x + 4), |x| < \frac{1}{4}. \end{aligned}$$

Theorem 1.16. *There exists the sum identity*

$$\begin{aligned} & \frac{1^8}{2^m} + \frac{2^8}{2^{2m}} + \cdots + \frac{n^8}{2^{mn}} + \cdots = \\ & = \frac{1}{(2^m - 1)^9} (2^m + 247 \cdot 2^{2m} + 4293 \cdot 2^{3m} + 15619 \cdot 2^{4m} + 15619 \cdot 2^{5m} + 4293 \cdot 2^{6m} + \\ & \quad + 247 \cdot 2^{7m} + 2^{8m}), m \in N_+. \end{aligned}$$

Theorem 1.17. *The generating function of series $\{4^n \cdot n^9\}$ is*

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} 4^n \cdot n^9 x^n = \\ &= \frac{x}{(1-4x)^{10}} (262144x^8 + 32899072x^7 + 239337472x^6 + 361406464x^5 + \\ & \quad + 159938560x^4 + 22587904x^3 + 934912x^2 + 8032x + 4), |x| < \frac{1}{4}. \end{aligned}$$

Theorem 1.18. *There exists the sum identity*

$$\begin{aligned} & \frac{1^9}{2^m} + \frac{2^9}{2^{2m}} + \cdots + \frac{n^9}{2^{mn}} + \cdots = \\ & = \frac{1}{(2^m - 1)^{10}} (2^m + 502 \cdot 2^{2m} + 14608 \cdot 2^{3m} + 88234 \cdot 2^{4m} + 156190 \cdot 2^{5m} + 88234 \cdot 2^{6m} + \\ & \quad + 14608 \cdot 2^{7m} + 502 \cdot 2^{8m} + 2^{9m}), m \in N_+. \end{aligned}$$

Theorem 1.19. *The generating function of series $\{4^n \cdot n^{10}\}$ is*

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} 4^n \cdot n^{10} x^n = \\ &= \frac{x}{(1-4x)^{11}} (1048576x^9 + 265551872x^8 + 3135242240x^7 + 7457865728x^6 + \\ & \quad + 5367209984x^5 + 1341802496x^4 + 116529152x^3 + 3061760x^2 + 16208x + 4), |x| < \frac{1}{4}. \end{aligned}$$

Theorem 1.20. *There exists the sum identity*

$$\begin{aligned} & \frac{1^{10}}{2^m} + \frac{2^{10}}{2^{2m}} + \cdots + \frac{n^{10}}{2^{mn}} + \cdots = \\ & = \frac{1}{(2^m - 1)^{11}} (2^m + 1013 \cdot 2^{2m} + 47840 \cdot 2^{3m} + 455192 \cdot 2^{4m} + 1310354 \cdot 2^{5m} + \\ & \quad + 1310354 \cdot 2^{6m} + 455192 \cdot 2^{7m} + 47840 \cdot 2^{8m} + 1013 \cdot 2^{9m} + 2^{10m}), m \in N_+. \end{aligned}$$

Theorem 1.21. *The generating function of series $\{4^n \cdot n^{11}\}$ is*

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} 4^n \cdot n^{11} x^n = \\ &= \frac{x}{(1-4x)^{12}} (4194304x^{10} + 2134900736x^9 + 37622906880x^8 + 119325851648x^7 + \\ & \quad + 159549259776x^6 + 64406519868x^5 + 9971828736x^4 + 101038080x^3 + 599696x^2 + \end{aligned}$$

$$+32576x + 4), |x| < \frac{1}{4}.$$

Theorem 1.22. *There exists the sum identity*

$$\begin{aligned} & \frac{1^{11}}{2^m} + \frac{2^{11}}{2^{2m}} + \cdots + \frac{n^{11}}{2^{mn}} + \cdots = \\ & = \frac{1}{(2^m - 1)^{12}} (2^m + 2036 \cdot 2^{2m} + 152637 \cdot 2^{3m} + 2203488 \cdot 2^{4m} + 9738114 \cdot 2^{5m} + \\ & + 15724248 \cdot 2^{6m} + 9738114 \cdot 2^{7m} + 2203488 \cdot 2^{8m} + 152637 \cdot 2^{9m} + 2036 \cdot 2^{10m} + 2^{11m}), \\ & m \in N_+. \end{aligned}$$

2. Proofs

Proof. (Theorem 1.1) For $1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, |x| < 1,$

$$1 + 4x + 4^2x^2 + \cdots + 4^n x^n + \cdots = \sum_{n=0}^{\infty} 4^n x^n = \sum_{n=0}^{\infty} (4x)^n = \frac{1}{1-4x},$$

$$|x| < \frac{1}{4}.$$

$$\begin{aligned} \left(\sum_{n=0}^{\infty} 4^n x^n \right)' &= \sum_{n=1}^{\infty} 4^n n x^{n-1} = \frac{1}{x} \sum_{n=1}^{\infty} 4^n n x^n \\ &= \frac{1}{x} \sum_{n=0}^{\infty} 4^n n x^n, \end{aligned}$$

and

$$\left(\frac{1}{1-4x} \right)' = \frac{4}{(1-4x)^2},$$

then

$$\frac{1}{x} \sum_{n=0}^{\infty} 4^n n x^n = \frac{4}{(1-4x)^2},$$

hence

$$\sum_{n=0}^{\infty} 4^n n x^n = \frac{4x}{(1-4x)^2}, |x| < \frac{1}{4}.$$

□

Proof. (Theorem 1.2) Let $x = \frac{1}{2^{m+2}}, m \in N_+.$ Then $|x| < \frac{1}{4},$ according to Theorem 1.1,

$$\begin{aligned} \sum_{n=0}^{\infty} 4^n n \left(\frac{1}{2^{m+2}} \right)^n &= \frac{4 \cdot \frac{1}{2^{m+2}}}{\left(1 - 4 \cdot \frac{1}{2^{m+2}} \right)^2}, \\ \sum_{n=0}^{\infty} n \cdot \frac{1}{2^{mn}} &= \frac{2^m}{(2^m - 1)^2}, \end{aligned}$$

namely, there exists the sum equation

$$\frac{1}{2^m} + \frac{2}{2^{2m}} + \cdots + \frac{n}{2^{mn}} + \cdots = \frac{2^m}{(2^m - 1)^2}, m \in N_+.$$

□

Specially, put $m=1$ in Theorem 1.2, we obtain the sum equality as follows:

$$\frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{n}{2^n} + \cdots = 2.$$

Proof. (Theorem 1.3) Analogous to Theorem 1.1,

$$\sum_{n=0}^{\infty} 4^n n x^n = \frac{4x}{(1-4x)^2}, |x| < \frac{1}{4}, (*)$$

Take derivatives on both sides of equation (*),

on the left,

$$\left(\sum_{n=0}^{\infty} 4^n n x^n\right)' = \sum_{n=1}^{\infty} 4^n n^2 x^{n-1} = \frac{1}{x} \sum_{n=1}^{\infty} 4^n n^2 x^n = \frac{1}{x} \sum_{n=0}^{\infty} 4^n n^2 x^n$$

on the right,

$$\left(\frac{4x}{(1-4x)^2}\right)' = \frac{4(1-4x)^2 - 4x \cdot 2(1-4x)(-4)}{(1-4x)^4} = \frac{16x+4}{(1-4x)^3},$$

so

$$\frac{1}{x} \sum_{n=0}^{\infty} 4^n n^2 x^n = \frac{16x+4}{(1-4x)^3},$$

namely,

$$\sum_{n=0}^{\infty} 4^n n^2 x^n = \frac{x(16x+4)}{(1-4x)^3}, |x| < \frac{1}{4}.$$

□

Proof. (Theorem 1.4)

Let $x = \frac{1}{2^{m+2}}$, $m \in N_+$. Then $|x| < \frac{1}{4}$, according to Theorem 1.3,

$$\sum_{n=0}^{\infty} 4^n n^2 \left(\frac{1}{2^{m+2}}\right)^n = \frac{1}{(1-4 \cdot \frac{1}{2^{m+2}})^3} \left[\frac{1}{2^{m+2}} (16 \cdot \frac{1}{2^{m+2}} + 4) \right],$$

$$\sum_{n=0}^{\infty} n^2 \cdot \frac{1}{2^{mn}} = \frac{2^m + 2^{2m}}{(2^m - 1)^3},$$

namely, there exists the sum identity as follows:

$$\frac{1^2}{2^m} + \frac{2^2}{2^{2m}} + \cdots + \frac{n^2}{2^{mn}} + \cdots = \frac{2^m + 2^{2m}}{(2^m - 1)^3}, m \in N_+.$$

Conclusions: the sequence is unimodal and symmetric ($a_k = a_{3-k}$, $1 \leq k \leq 2$), with two peaks: $\frac{1}{(2^m-1)^3}$, $\frac{1}{(2^m-1)^3}$, also $a_1 = a_2$.

□

Specially, put $m=1$ in Theorem 1.4, the sum equality is derived as follows:

$$\frac{1^2}{2} + \frac{2^2}{2^2} + \cdots + \frac{n^2}{2^n} + \cdots = \frac{2^1 + 2^{2 \times 1}}{(2^1 - 1)^3} = 6.$$

Proof. (Theorem 1.5) Analogous to Theorem 1.3,

$$\sum_{n=0}^{\infty} 4^n n^2 x^n = \frac{x(16x+4)}{(1-4x)^3}, |x| < \frac{1}{4}, (**)$$

Take derivatives on both sides of equation (**),
on the left,

$$\left(\sum_{n=0}^{\infty} 4^n n^2 x^n\right)' = \sum_{n=1}^{\infty} 4^n n^3 x^{n-1} = \frac{1}{x} \sum_{n=1}^{\infty} 4^n n^3 x^n = \frac{1}{x} \sum_{n=0}^{\infty} 4^n n^3 x^n$$

on the right,

$$\left[\frac{x(16x+4)}{(1-4x)^3}\right]' = \frac{64x^2 + 64x + 4}{(1-4x)^4},$$

(To save space, detailed calculations are omitted)

hence

$$\frac{1}{x} \sum_{n=0}^{\infty} 4^n n^3 x^n = \frac{64x^2 + 64x + 4}{(1-4x)^4},$$

finally ,

$$\sum_{n=0}^{\infty} 4^n n^3 x^n = \frac{x(64x^2 + 64x + 4)}{(1-4x)^4}, |x| < \frac{1}{4}.$$

□

Proof. (Theorem 1.6) Let $x = \frac{1}{2^{m+2}}$, $m \in N_+$. Then $|x| < \frac{1}{4}$, according to Theorem 1.5,

$$\begin{aligned} \sum_{n=0}^{\infty} 4^n n^3 \left(\frac{1}{2^{m+2}}\right)^n &= \frac{1}{(1-4 \cdot \frac{1}{2^{m+2}})^4} \cdot \frac{1}{2^{m+2}} [64 \cdot \left(\frac{1}{2^{m+2}}\right)^2 + 64 \cdot \frac{1}{2^{m+2}} + 4], \\ \sum_{n=0}^{\infty} n^3 \cdot \frac{1}{2^{mn}} &= \frac{2^m + 4 \cdot 2^{2m} + 2^{3m}}{(2^m - 1)^4}, \end{aligned}$$

(Due to space constraints, detailed calculations have been omitted)

so, there is the sum identity

$$\frac{1^3}{2^m} + \frac{2^3}{2^{2m}} + \cdots + \frac{n^3}{2^{mn}} + \cdots = \frac{2^m + 4 \cdot 2^{2m} + 2^{3m}}{(2^m - 1)^4}, m \in N_+.$$

Conclusions: the sequence is unimodal and symmetric ($a_k = a_{4-k}$, $1 \leq k \leq 3$),
with one peak: $\frac{4}{(2^m - 1)^4}$. □

In particular, put $m=1$ in Theorem 1.6, the sum equality is derived as follows:

$$\frac{1^3}{2} + \frac{2^3}{2^2} + \cdots + \frac{n^3}{2^n} + \cdots = \frac{2^1 + 4 \cdot 2^{2 \times 1} + 2^{3 \times 1}}{(2^1 - 1)^4} = 26.$$

Proof. (Theorem 1.7) Analogous to Theorem 1.5,

$$\sum_{n=0}^{\infty} 4^n n^3 x^n = \frac{x(64x^2 + 64x + 4)}{(1-4x)^4}, |x| < \frac{1}{4}, (***)$$

Take derivatives on both sides of equation (**),

on the left,

$$\left(\sum_{n=0}^{\infty} 4^n n^3 x^n\right)' = \sum_{n=1}^{\infty} 4^n n^4 x^{n-1} = \frac{1}{x} \sum_{n=1}^{\infty} 4^n n^4 x^n = \frac{1}{x} \sum_{n=0}^{\infty} 4^n n^4 x^n$$

on the right,

$$\left[\frac{x(64x^2 + 64x + 4)}{(1-4x)^4}\right]' = \frac{256x^3 + 704x^2 + 176x + 4}{(1-4x)^5},$$

so

$$\frac{1}{x} \sum_{n=0}^{\infty} 4^n n^4 x^n = \frac{256x^3 + 704x^2 + 176x + 4}{(1-4x)^5}$$

namely, the generating function of series $\{4^n \cdot n^4\}$ is

$$f(x) = \sum_{n=0}^{\infty} 4^n \cdot n^4 x^n = \frac{x(256x^3 + 704x^2 + 176x + 4)}{(1-4x)^5}, |x| < \frac{1}{4}.$$

□

Proof. (Theorem 1.8) Let $x = \frac{1}{2^{m+2}}$, $m \in N_+$. Then $|x| < \frac{1}{4}$, according to Theorem 1.7,

$$\begin{aligned} \sum_{n=0}^{\infty} 4^n n^4 \left(\frac{1}{2^{m+2}}\right)^n &= \frac{1}{(1-4 \cdot \frac{1}{2^{m+2}})^5} \cdot \frac{1}{2^{m+2}} [256 \cdot \left(\frac{1}{2^{m+2}}\right)^3 + 704 \cdot \left(\frac{1}{2^{m+2}}\right)^2 + \\ &\quad + 176 \cdot \frac{1}{2^{m+2}} + 4], \\ \sum_{n=0}^{\infty} n^4 \cdot \frac{1}{2^{mn}} &= \frac{2^m + 11 \cdot 2^{2m} + 11 \cdot 2^{3m} + 2^{4m}}{(2^m - 1)^5}. \end{aligned}$$

Then there exists the sum identity

$$\frac{1^4}{2^m} + \frac{2^4}{2^{2m}} + \cdots + \frac{n^4}{2^{mn}} + \cdots = \frac{2^m + 11 \cdot 2^{2m} + 11 \cdot 2^{3m} + 2^{4m}}{(2^m - 1)^5}, m \in N_+.$$

Conclusions: the sequence is unimodal and symmetric ($a_k = a_{5-k}$, $1 \leq k \leq 4$), with two peaks: $\frac{11}{(2^m-1)^5}$ and $\frac{11}{(2^m-1)^5}$. □

In particular, put $m=1$ in Theorem 1.7, the sum equality is derived as follows:

$$\frac{1^4}{2} + \frac{2^4}{2^2} + \cdots + \frac{n^4}{2^n} + \cdots = \frac{2^1 + 4 \cdot 2^{2 \times 1} + 4 \cdot 2^{3 \times 1} + 2^{4 \times 1}}{(2^1 - 1)^4} = 150.$$

Proof. (Theorem 1.9) Analogous to Theorem 1.7,

$$\sum_{n=0}^{\infty} 4^n n^4 x^n = \frac{x(256x^3 + 704x^2 + 176x + 4)}{(1-4x)^5}, |x| < \frac{1}{4}, (***)$$

Take derivatives on both sides of equation (***),

on the left,

$$\left(\sum_{n=0}^{\infty} 4^n n^4 x^n\right)' = \sum_{n=1}^{\infty} 4^n n^5 x^{n-1} = \frac{1}{x} \sum_{n=1}^{\infty} 4^n n^5 x^n = \frac{1}{x} \sum_{n=0}^{\infty} 4^n n^5 x^n$$

on the right,

$$\left[\frac{x(256x^3 + 704x^2 + 176x + 4)}{(1-4x)^5} \right]' = \frac{1024x^4 + 6656x^3 + 4224x^2 + 416x + 4}{(1-4x)^6},$$

so

$$\begin{aligned} \frac{1}{x} \sum_{n=0}^{\infty} 4^n n^5 x^n &= \frac{1024x^4 + 6656x^3 + 4224x^2 + 416x + 4}{(1-4x)^6} \\ \sum_{n=0}^{\infty} 4^n n^5 x^n &= \frac{x(1024x^4 + 6656x^3 + 4224x^2 + 416x + 4)}{(1-4x)^6}, |x| < \frac{1}{4}. \end{aligned}$$

□

Proof. (Theorem 1.10) Let $x = \frac{1}{2^{m+2}}$, $m \in N_+$. Then $|x| < \frac{1}{4}$, according to Theorem 1.9,

$$\begin{aligned} \sum_{n=0}^{\infty} 4^n n^5 \left(\frac{1}{2^{m+2}}\right)^n &= \frac{1}{(1-4 \cdot \frac{1}{2^{m+2}})^6} \cdot \frac{1}{2^{m+2}} [1024 \cdot \left(\frac{1}{2^{m+2}}\right)^4 + 6656 \cdot \left(\frac{1}{2^{m+2}}\right)^3 + \\ &\quad + 4224 \cdot \left(\frac{1}{2^{m+2}}\right)^2 + 416 \cdot \frac{1}{2^{m+2}} + 4], \\ \sum_{n=0}^{\infty} n^5 \cdot \frac{1}{2^{mn}} &= \frac{2^m + 26 \cdot 2^{2m} + 66 \cdot 2^{3m} + 26 \cdot 2^{4m} + 2^{5m}}{(2^m - 1)^6}. \end{aligned}$$

Then there exists the sum identity

$$\frac{1^5}{2^m} + \frac{2^5}{2^{2m}} + \cdots + \frac{n^5}{2^{mn}} + \cdots = \frac{2^m + 26 \cdot 2^{2m} + 66 \cdot 2^{3m} + 26 \cdot 2^{4m} + 2^{5m}}{(2^m - 1)^6}, m \in N_+.$$

Conclusions: the sequence is unimodal and symmetric ($a_k = a_{6-k}$, $1 \leq k \leq 5$), with one peak: $\frac{66}{(2^m - 1)^6}$. □

In particular, put $m=1$ in Theorem 1.10, the sum equality is derived as follows:

$$\frac{1^5}{2} + \frac{2^5}{2^2} + \cdots + \frac{n^5}{2^n} + \cdots = \frac{2^1 + 26 \cdot 2^{2 \times 1} + 66 \cdot 2^{3 \times 1} + 26 \cdot 2^{4 \times 1} + 2^{5 \times 1}}{(2^1 - 1)^6} = 1082.$$

Proof. (Theorem 1.11) Analogous to Theorem 1.9,

$$\sum_{n=0}^{\infty} 4^n n^5 x^n = \frac{x(1024x^4 + 6656x^3 + 4224x^2 + 416x + 4)}{(1-4x)^6}, |x| < \frac{1}{4}, (** ** *)$$

Take derivatives on both sides of equation (** ** *),
on the left,

$$\left(\sum_{n=0}^{\infty} 4^n n^5 x^n \right)' = \sum_{n=1}^{\infty} 4^n n^6 x^{n-1} = \frac{1}{x} \sum_{n=1}^{\infty} 4^n n^6 x^n = \frac{1}{x} \sum_{n=0}^{\infty} 4^n n^6 x^n$$

on the right,

$$\begin{aligned} \left[\frac{x(1024x^4 + 6656x^3 + 4224x^2 + 416x + 4)}{(1-4x)^6} \right]' &= \frac{1}{(1-4x)^7} (4096x^5 + 58368x^4 + \\ &\quad + 77312x^3 + 19328x^2 + 912x + 4), \end{aligned}$$

hence

$$\begin{aligned} \frac{1}{x} \sum_{n=0}^{\infty} 4^n n^6 x^n &= \frac{1}{(1-4x)^7} (4096x^5 + 58368x^4 + 77312x^3 + 19328x^2 + 912x + 4), \\ \sum_{n=0}^{\infty} 4^n \cdot n^6 x^n &= \frac{x}{(1-4x)^7} (4096x^5 + 58368x^4 + \\ &+ 77312x^3 + 19328x^2 + 912x + 4), |x| < \frac{1}{4}. \end{aligned}$$

□

Proof. (Theorem 1.12) By setting $x = \frac{1}{2^{m+2}}$, $m \in N_+$. Then $|x| < \frac{1}{4}$, according to Theorem 1.11,

$$\begin{aligned} \sum_{n=0}^{\infty} 4^n n^6 \left(\frac{1}{2^{m+2}}\right)^n &= \frac{1}{(1-4 \cdot \frac{1}{2^{m+2}})^7} \cdot \frac{1}{2^{m+2}} [4096 \cdot \left(\frac{1}{2^{m+2}}\right)^5 + 58368 \cdot \left(\frac{1}{2^{m+2}}\right)^4 + \\ &+ 77312 \cdot \left(\frac{1}{2^{m+2}}\right)^3 + 19328 \cdot \left(\frac{1}{2^{m+2}}\right)^2 + 912 \cdot \frac{1}{2^{m+2}} + 4], \\ \sum_{n=0}^{\infty} n^6 \cdot \frac{1}{2^{mn}} &= \frac{2^m + 57 \cdot 2^{2m} + 302 \cdot 2^{3m} + 302 \cdot 2^{4m} + 57 \cdot 2^{5m} + 2^{6m}}{(2^m - 1)^7}. \end{aligned}$$

Namely there exists the sum identity

$$\begin{aligned} \frac{1^6}{2^m} + \frac{2^6}{2^{2m}} + \cdots + \frac{n^6}{2^{mn}} + \cdots &= \\ = \frac{2^m + 57 \cdot 2^{2m} + 302 \cdot 2^{3m} + 302 \cdot 2^{4m} + 57 \cdot 2^{5m} + 2^{6m}}{(2^m - 1)^7}, m \in N_+. \end{aligned}$$

Conclusions: (1) base $2^m, 2^{2m}, 2^{3m}, 2^{4m}, 2^{5m}, 2^{6m}$

(2) the sequence of coefficients is unimodal and symmetric ($a_k = a_{7-k}, 1 \leq k \leq 6$)

(3) with two peaks, $\frac{302}{(2^m-1)^7}$ and $\frac{302}{(2^{m-1})^7}$, also $a_3 = a_4$ □

In particular, put $m=1$ in Theorem 1.12, the sum equality is derived as follows:

$$\begin{aligned} \frac{1^6}{2} + \frac{2^6}{2^2} + \cdots + \frac{n^6}{2^n} + \cdots &= \frac{2^1 + 57 \cdot 2^{2 \times 1} + 302 \cdot 2^{3 \times 1} + 302 \cdot 2^{4 \times 1} + 57 \cdot 2^{5 \times 1} + 2^{6 \times 1}}{(2^1 - 1)^7} \\ &= 9366. \end{aligned}$$

Proof. (Theorem 1.13) Analogous to Theorem 1.11,

$$\sum_{n=0}^{\infty} 4^n \cdot n^6 x^n = \frac{x}{(1-4x)^7} (4096x^5 + 58368x^4 + 77312x^3 + 19328x^2 + 912x + 4), |x| < \frac{1}{4}.$$

(***)

Take derivatives on both sides of equation (***) ,
on the left,

$$\left(\sum_{n=0}^{\infty} 4^n n^6 x^n\right)' = \sum_{n=1}^{\infty} 4^n n^7 x^{n-1} = \frac{1}{x} \sum_{n=1}^{\infty} 4^n n^7 x^n = \frac{1}{x} \sum_{n=0}^{\infty} 4^n n^7 x^n,$$

on the right,

$$\left[\frac{x(4096x^5 + 58368x^4 + 77312x^3 + 19328x^2 + 912x + 4)}{(1-4x)^6} \right]' = \frac{1}{(1-4x)^8} (16384x^6 + 491520x^5 + 1219584x^4 + 6184968x^3 + 76224x^2 + 1920x + 4),$$

so

$$\frac{1}{x} \sum_{n=0}^{\infty} 4^n n^7 x^n = \frac{1}{(1-4x)^8} (16384x^6 +$$

$$491520x^5 + 1219584x^4 + 6184968x^3 + 76224x^2 + 1920x + 4),$$

finally, the generating function of series $\{4^n \cdot n^7\}$ is

$$f(x) = \sum_{n=0}^{\infty} 4^n \cdot n^7 x^n =$$

$$= \frac{x}{(1-4x)^8} (16384x^6 + 491520x^5 + 1219584x^4 + 618496x^3 + 76224x^2 + 1920x + 4),$$

$|x| < \frac{1}{4}$. □

Proof. (Theorem 1.14) By setting $x = \frac{1}{2^{m+2}}$, $m \in N_+$. Then $|x| < \frac{1}{4}$, according to Theorem 1.13,

$$\begin{aligned} \sum_{n=0}^{\infty} 4^n n^7 \left(\frac{1}{2^{m+2}}\right)^n &= \frac{1}{(1-4 \cdot \frac{1}{2^{m+2}})^8} \cdot \frac{1}{2^{m+2}} [16384 \cdot \left(\frac{1}{2^{m+2}}\right)^6 + 491520 \cdot \left(\frac{1}{2^{m+2}}\right)^5 + \\ &+ 1219584 \cdot \left(\frac{1}{2^{m+2}}\right)^4 + 618496 \cdot \left(\frac{1}{2^{m+2}}\right)^3 + 76224 \cdot \left(\frac{1}{2^{m+2}}\right)^2 + 1920 \cdot \frac{1}{2^{m+2}} + 4], \\ \sum_{n=0}^{\infty} n^7 \cdot \frac{1}{2^{mn}} &= \frac{1}{(2^m - 1)^8} (2^m + 120 \cdot 2^{2m} + 1191 \cdot 2^{3m} + 2416 \cdot 2^{4m} + 1191 \cdot 2^{5m} + \\ &+ 120 \cdot 2^{6m} + 2^{7m}). \end{aligned}$$

Then there exists the sum identity

$$\begin{aligned} \frac{1^7}{2^m} + \frac{2^7}{2^{2m}} + \cdots + \frac{n^7}{2^{mn}} + \cdots &= \\ = \frac{2^m + 120 \cdot 2^{2m} + 1191 \cdot 2^{3m} + 2416 \cdot 2^{4m} + 1191 \cdot 2^{5m} + 120 \cdot 2^{6m} + 2^{7m}}{(2^m - 1)^8}, \end{aligned}$$

$m \in N_+$.

Conclusions: (1) base $2^m, 2^{2m}, 2^{3m}, 2^{4m}, 2^{5m}, 2^{6m}, 2^{7m}$

(2) the sequence of coefficients is unimodal and symmetric ($a_k = a_{8-k}, 1 \leq k \leq 7$)

(3) with one peak, $\frac{2416}{(2^m-1)^8}$ □

Note That it is not difficult to see the complexity of the calculation in proving course here.

In particular, set $m=1$ in Theorem 1.14, the sum equality is derived as follows:

$$\frac{1^7}{2} + \frac{2^7}{2^2} + \cdots + \frac{n^7}{2^n} + \cdots = 94586.$$

Proof. (Theorem 1.15) Analogous to Theorem 1.13,

$$\begin{aligned} & \sum_{n=0}^{\infty} 4^n \cdot n^7 x^n = \\ & = \frac{x}{(1-4x)^8} (16384x^6 + 491520x^5 + 1219584x^4 + 618496x^3 + 76224x^2 + 1920x + 4), \\ & |x| < \frac{1}{4}. (** ** ** ** *) \end{aligned}$$

Take derivatives on both sides of equation (** ** ** ** *),
on the left,

$$\left(\sum_{n=0}^{\infty} 4^n n^7 x^n \right)' = \sum_{n=1}^{\infty} 4^n n^8 x^{n-1} = \frac{1}{x} \sum_{n=1}^{\infty} 4^n n^8 x^n = \frac{1}{x} \sum_{n=0}^{\infty} 4^n n^8 x^n,$$

on the right,

$$\begin{aligned} & \left[\frac{x}{(1-4x)^8} (16384x^6 + 491520x^5 + 1219584x^4 + 618496x^3 + 76224x^2 + 1920x + 4) \right]' = \\ & = \frac{1}{(1-4x)^9} (65536x^7 + 4046848x^6 + 17584128x^5 + 15993856x^4 + 3998464x^3 + \\ & \quad + 274752x^2 + 3952x + 4), \end{aligned}$$

so

$$\begin{aligned} \frac{1}{x} \sum_{n=0}^{\infty} 4^n n^8 x^n & = \frac{1}{(1-4x)^9} (65536x^7 + 4046848x^6 + 17584128x^5 + 15993856x^4 + \\ & \quad + 3998464x^3 + 274752x^2 + 3952x + 4), \end{aligned}$$

finally, the generating function of series $\{4^n \cdot n^8\}$ is

$$\begin{aligned} f(x) & = \sum_{n=0}^{\infty} 4^n \cdot n^8 x^n = \\ & = \frac{x}{(1-4x)^9} (65536x^7 + 4046848x^6 + 17584128x^5 + 15993856x^4 + 3998464x^3 + \\ & \quad + 274752x^2 + 3952x + 4), |x| < \frac{1}{4}. \end{aligned}$$

□

Proof. (Theorem 1.16) By setting $x = \frac{1}{2^{m+2}}$, $m \in N_+$. Then $|x| < \frac{1}{4}$, according to Theorem 1.15,

$$\begin{aligned} & \sum_{n=0}^{\infty} 4^n n^8 \left(\frac{1}{2^{m+2}}\right)^n = \frac{1}{(1-4 \cdot \frac{1}{2^{m+2}})^9} \cdot \frac{1}{2^{m+2}} [65536 \cdot \left(\frac{1}{2^{m+2}}\right)^7 + \\ & \quad + 4046848 \cdot \left(\frac{1}{2^{m+2}}\right)^6 + 17584128 \cdot \left(\frac{1}{2^{m+2}}\right)^5 + \\ & \quad + 15993856 \cdot \left(\frac{1}{2^{m+2}}\right)^4 + 3998464 \cdot \left(\frac{1}{2^{m+2}}\right)^3 + 274752 \cdot \left(\frac{1}{2^{m+2}}\right)^2 + 3952 \cdot \frac{1}{2^{m+2}} + 4], \\ & \sum_{n=0}^{\infty} n^8 \cdot \frac{1}{2^{mn}} = \frac{1}{(2^m - 1)^9} (2^m + 247 \cdot 2^{2m} + 4293 \cdot 2^{3m} + 15619 \cdot 2^{4m} + \\ & \quad + 15619 \cdot 2^{5m} + 4293 \cdot 2^{6m} + 247 \cdot 2^{7m} + 2^{8m}). \end{aligned}$$

Then there exists the sum identity

$$\begin{aligned} & \frac{1^8}{2^m} + \frac{2^8}{2^{2m}} + \cdots + \frac{n^8}{2^{mn}} + \cdots = \\ = & \frac{1}{(2^m - 1)^9} (2^m + 247 \cdot 2^{2m} + 4293 \cdot 2^{3m} + 15619 \cdot 2^{4m} + 15619 \cdot 2^{5m} + 4293 \cdot 2^{6m} + \\ & + 247 \cdot 2^{7m} + 2^{8m}), m \in N_+. \end{aligned}$$

Conclusions:

- (1) base $2^m, 2^{2m}, 2^{3m}, 2^{4m}, 2^{5m}, 2^{6m}, 2^{7m}, 2^{8m}$
- (2) the sequence of coefficients is unimodal and symmetric ($a_k = a_{9-k}, 1 \leq k \leq 8$)
- (3) with two peaks, $\frac{15619}{(2^m-1)^9}$ and $\frac{15619}{(2^m-1)^9}$, also $a_4 = a_5$ \square

In particular, set $m=1$ in Theorem 1.16, the sum equality is derived as follows:

$$\frac{1^8}{2} + \frac{2^8}{2^2} + \cdots + \frac{n^8}{2^n} + \cdots = 1091670.$$

Note That requires strong computing power and skills.

Proof. (Theorem 1.17) Analogous to Theorem 1.15,

$$\begin{aligned} & \sum_{n=0}^{\infty} 4^n \cdot n^8 x^n = \\ = & \frac{x}{(1-4x)^9} (65536x^7 + 4046848x^6 + 17584128x^5 + 15993856x^4 + 3998464x^3 + \\ & + 274752x^2 + 3952x + 4), |x| < \frac{1}{4}. (***) \end{aligned}$$

Take derivatives on both sides of equation (***)
on the left,

$$\left(\sum_{n=0}^{\infty} 4^n n^8 x^n \right)' = \sum_{n=1}^{\infty} 4^n n^9 x^{n-1} = \frac{1}{x} \sum_{n=1}^{\infty} 4^n n^9 x^n = \frac{1}{x} \sum_{n=0}^{\infty} 4^n n^9 x^n,$$

on the right,

$$\begin{aligned} & \left[\frac{x}{(1-4x)^9} (65536x^7 + 4046848x^6 + 17584128x^5 + 15993856x^4 + 3998464x^3 + \right. \\ & \quad \left. + 274752x^2 + 3952x + 4) \right]' = \\ = & \frac{1}{(1-4x)^{10}} (262144x^8 + 32899072x^7 + 239337472x^6 + 361406464x^5 + \\ & + 159938560x^4 + 22587904x^3 + 934912x^2 + 8032x + 4), \end{aligned}$$

hence

$$\begin{aligned} \frac{1}{x} \sum_{n=0}^{\infty} 4^n n^9 x^n &= \frac{1}{(1-4x)^{10}} (262144x^8 + 32899072x^7 + 239337472x^6 + 361406464x^5 + \\ & + 159938560x^4 + 22587904x^3 + 934912x^2 + 8032x + 4), \end{aligned}$$

on the end, the generating function of the sequence $\{4^n \cdot n^9\}$ is

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} 4^n \cdot n^9 x^n = \\ &= \frac{x}{(1-4x)^{10}} (262144x^8 + 32899072x^7 + 239337472x^6 + 361406464x^5 + \\ &\quad + 159938560x^4 + 22587904x^3 + 934912x^2 + 8032x + 4), |x| < \frac{1}{4}. \end{aligned}$$

□

Proof. (Theorem 1.18) By setting $x = \frac{1}{2^{m+2}}$, $m \in N_+$. Then $|x| < \frac{1}{4}$, according to Theorem 1.17,

$$\begin{aligned} \sum_{n=0}^{\infty} 4^n n^9 \left(\frac{1}{2^{m+2}}\right)^n &= \frac{1}{(1-4 \cdot \frac{1}{2^{m+2}})^{10}} \cdot \frac{1}{2^{m+2}} [262144 \cdot \left(\frac{1}{2^{m+2}}\right)^8 + 32899072 \cdot \left(\frac{1}{2^{m+2}}\right)^7 \\ &\quad + 239337472 \cdot \left(\frac{1}{2^{m+2}}\right)^6 + 361406464 \cdot \left(\frac{1}{2^{m+2}}\right)^5 + \\ &\quad + 159938560 \cdot \left(\frac{1}{2^{m+2}}\right)^4 + 22587904 \cdot \left(\frac{1}{2^{m+2}}\right)^3 + 934912 \cdot \left(\frac{1}{2^{m+2}}\right)^2 + 8032 \cdot \frac{1}{2^{m+2}} + 4], \\ \sum_{n=0}^{\infty} n^9 \cdot \frac{1}{2^{mn}} &= \frac{1}{(2^m-1)^{10}} (2^m + 502 \cdot 2^{2m} + 14608 \cdot 2^{3m} + 88234 \cdot 2^{4m} + \\ &\quad + 156190 \cdot 2^{5m} + 88234 \cdot 2^{6m} + 14608 \cdot 2^{7m} + 502 \cdot 2^{8m} + 2^{9m}). \end{aligned}$$

Namely there exists the sum identity

$$\begin{aligned} &\frac{1^9}{2^m} + \frac{2^9}{2^{2m}} + \cdots + \frac{n^9}{2^{mn}} + \cdots = \\ &= \frac{1}{(2^m-1)^{10}} (2^m + 502 \cdot 2^{2m} + 14608 \cdot 2^{3m} + 88234 \cdot 2^{4m} + 156190 \cdot 2^{5m} + 88234 \cdot 2^{6m} + \\ &\quad + 14608 \cdot 2^{7m} + 502 \cdot 2^{8m} + 2^{9m}), m \in N_+. \end{aligned}$$

Conclusions:

- (1) base $2^m, 2^{2m}, 2^{3m}, 2^{4m}, 2^{5m}, 2^{6m}, 2^{7m}, 2^{8m}, 2^{9m}$
- (2) the sequence of coefficients is unimodal and symmetric ($a_k = a_{10-k}, 1 \leq k \leq 9$)
- (3) with one peak, $\frac{156190}{(2^m-1)^{10}}$ □

In particular, set $m=1$ in Theorem 1.18, the sum equality is derived as follows:

$$\frac{1^9}{2} + \frac{2^9}{2^2} + \cdots + \frac{n^9}{2^n} + \cdots = 14174522.$$

Proof. (Theorem 1.19) Analogous to Theorem 1.17,

$$\begin{aligned} &\sum_{n=0}^{\infty} 4^n \cdot n^9 x^n = \\ &= \frac{x}{(1-4x)^{10}} (262144x^8 + 32899072x^7 + 239337472x^6 + 361406464x^5 + \\ &\quad + 159938560x^4 + 22587904x^3 + 934912x^2 + 8032x + 4), |x| < \frac{1}{4}. (***) \end{aligned}$$

Take derivatives on both sides of equation (*****),
on the left,

$$\left(\sum_{n=0}^{\infty} 4^n n^9 x^n\right)' = \sum_{n=1}^{\infty} 4^n n^{10} x^{n-1} = \frac{1}{x} \sum_{n=1}^{\infty} 4^n n^{10} x^n = \frac{1}{x} \sum_{n=0}^{\infty} 4^n n^{10} x^n,$$

on the right,

$$\begin{aligned} & \left[\frac{x}{(1-4x)^{10}} (262144x^8 + 32899072x^7 + 239337472x^6 + 361406464x^5 + 159938560x^4 + \right. \\ & \quad \left. + 22587904x^3 + 934912x^2 + 8032x + 4) \right]' = \\ & = \frac{1}{(1-4x)^{11}} (1048576x^9 + 265551872x^8 + 3135242240x^7 + 7457865728x^6 + \\ & \quad + 5367209984x^5 + 1341802496x^4 + 116529152x^3 + 3061760x^2 + 16208x + 4). \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{x} \sum_{n=0}^{\infty} 4^n n^{10} x^n &= \frac{1}{(1-4x)^{11}} (1048576x^9 + 265551872x^8 + 3135242240x^7 + \\ & \quad + 7457865728x^6 + 5367209984x^5 + 1341802496x^4 + 116529152x^3 + 3061760x^2 + \\ & \quad + 16208x + 4) \end{aligned}$$

The generating function of the sequence $\{4^n \cdot n^{10}\}$ is

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} 4^n \cdot n^{10} x^n = \\ &= \frac{x}{(1-4x)^{11}} (1048576x^9 + 265551872x^8 + 3135242240x^7 + 7457865728x^6 + \\ & \quad + 5367209984x^5 + 1341802496x^4 + 116529152x^3 + 3061760x^2 + 16208x + 4), |x| < \frac{1}{4}. \end{aligned}$$

□

Proof. (Theorem 1.20) By setting $x = \frac{1}{2^{m+2}}$, $m \in N_+$. Then $|x| < \frac{1}{4}$, according to Theorem 1.19,

$$\begin{aligned} \sum_{n=0}^{\infty} 4^n n^{10} \left(\frac{1}{2^{m+2}}\right)^n &= \frac{1}{(1-4 \cdot \frac{1}{2^{m+2}})^{11}} \cdot \frac{1}{2^{m+2}} [1048576 \cdot \left(\frac{1}{2^{m+2}}\right)^9 + \\ & \quad + 265551872 \cdot \left(\frac{1}{2^{m+2}}\right)^8 + 3135242240 \cdot \left(\frac{1}{2^{m+2}}\right)^7 + 7457865728 \cdot \left(\frac{1}{2^{m+2}}\right)^6 + \\ & \quad + 5367209984 \cdot \left(\frac{1}{2^{m+2}}\right)^5 + 1341802496 \cdot \left(\frac{1}{2^{m+2}}\right)^4 + 116529152 \cdot \left(\frac{1}{2^{m+2}}\right)^3 + \\ & \quad + 3061760 \cdot \left(\frac{1}{2^{m+2}}\right)^2 + 16208 \cdot \frac{1}{2^{m+2}} + 4], \\ \sum_{n=0}^{\infty} n^{10} \cdot \frac{1}{2^{mn}} &= \frac{1}{(2^m - 1)^{11}} (2^m + 1013 \cdot 2^{2m} + 47840 \cdot 2^{3m} + 455192 \cdot 2^{4m} + \\ & \quad + 1310354 \cdot 2^{5m} + 1310354 \cdot 2^{6m} + 455192 \cdot 2^{7m} + 47840 \cdot 2^{8m} + 1013 \cdot 2^{9m} + 2^{10m}). \end{aligned}$$

Namely, there exists the sum identity

$$\begin{aligned} & \frac{1^{10}}{2^m} + \frac{2^{10}}{2^{2m}} + \cdots + \frac{n^{10}}{2^{mn}} + \cdots = \\ & = \frac{1}{(2^m - 1)^{11}} (2^m + 1013 \cdot 2^{2m} + 47840 \cdot 2^{3m} + 455192 \cdot 2^{4m} + 1310354 \cdot 2^{5m} + \\ & + 1310354 \cdot 2^{6m} + 455192 \cdot 2^{7m} + 47840 \cdot 2^{8m} + 1013 \cdot 2^{9m} + 2^{10m}), m \in N_+. \end{aligned}$$

Conclusions:

- (1) base $2^m, 2^{2m}, 2^{3m}, 2^{4m}, 2^{5m}, 2^{6m}, 2^{7m}, 2^{8m}, 2^{9m}, 2^{10m}$
- (2) the sequence of coefficients is unimodal and symmetric ($a_k = a_{11-k}, 1 \leq k \leq 10$)
- (3) with one peaks, $\frac{1310354}{(2^m - 1)^{11}}$ and $\frac{1310354}{(2^m - 1)^{11}}$, also $a_5 = a_6$. □

In particular, set $m=1$ in Theorem 1.20, the sum equality is derived as follows:

$$\begin{aligned} & \frac{1^{10}}{2} + \frac{2^{10}}{2^2} + \cdots + \frac{n^{10}}{2^n} + \cdots = \\ & = \frac{1}{(2^1 - 1)^{11}} (2^1 + 1013 \cdot 2^{2 \times 1} + 47840 \cdot 2^{3 \times 1} + 455192 \cdot 2^{4 \times 1} + 1310354 \cdot 2^{5 \times 1} + \\ & + 1310354 \cdot 2^{6 \times 1} + 455192 \cdot 2^{7 \times 1} + 47840 \cdot 2^{8 \times 1} + 1013 \cdot 2^{9 \times 1} + 2^{10 \times 1}) = 204495126. \end{aligned}$$

Proof. (Theorem 1.21) Analogous to Theorem 1.19,

$$\begin{aligned} & \sum_{n=0}^{\infty} 4^n \cdot n^{10} x^n = \\ & = \frac{x}{(1 - 4x)^{11}} (1048576x^9 + 265551872x^8 + 3135242240x^7 + 7457865728x^6 + \\ & + 5367209984x^5 + 1341802496x^4 + 116529152x^3 + 3061760x^2 + 16208x + 4), |x| < \frac{1}{4}. \end{aligned}$$

(*****)

Take derivatives on both sides of equation (*****),
on the left,

$$\left(\sum_{n=0}^{\infty} 4^n n^{10} x^n \right)' = \sum_{n=1}^{\infty} 4^n n^{11} x^{n-1} = \frac{1}{x} \sum_{n=1}^{\infty} 4^n n^{11} x^n = \frac{1}{x} \sum_{n=0}^{\infty} 4^n n^{11} x^n,$$

on the right,

$$\begin{aligned} & \left[\frac{x}{(1 - 4x)^{11}} (1048576x^9 + 265551872x^8 + 3135242240x^7 + 7457865728x^6 + \right. \\ & \left. + 5367209984x^5 + 1341802496x^4 + 116529152x^3 + 3061760x^2 + 16208x + 4) \right]' = \\ & = \frac{1}{(1 - 4x)^{12}} (4194304x^{10} + 2134900736x^9 + 37622906880x^8 + 119325851648x^7 + \\ & + 159549259776x^6 + 64406519868x^5 + 9971828736x^4 + 101038080x^3 + 599696x^2 + \\ & + 32576x + 4). \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{x} \sum_{n=0}^{\infty} 4^n n^{11} x^n &= \frac{1}{(1-4x)^{12}} (4194304x^{10} + 2134900736x^9 + 37622906880x^8 + \\ &+ 119325851648x^7 + 159549259776x^6 + 64406519868x^5 + 9971828736x^4 + \\ &+ 101038080x^3 + 599696x^2 + 32576x + 4) \end{aligned}$$

Namely, the generating function of the sequence $\{4^n \cdot n^{11}\}$ is

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} 4^n \cdot n^{11} x^n = \\ &= \frac{x}{(1-4x)^{12}} (4194304x^{10} + 2134900736x^9 + 37622906880x^8 + 119325851648x^7 + \\ &+ 159549259776x^6 + 64406519868x^5 + 9971828736x^4 + 101038080x^3 + 599696x^2 + \\ &+ 32576x + 4), |x| < \frac{1}{4}. \end{aligned}$$

□

Proof. (Theorem 1.22) By setting $x = \frac{1}{2^{m+2}}$, $m \in N_+$. Then $|x| < \frac{1}{4}$, according to Theorem 1.21,

$$\begin{aligned} \sum_{n=0}^{\infty} 4^n n^{11} \left(\frac{1}{2^{m+2}}\right)^n &= \frac{1}{(1-4 \cdot \frac{1}{2^{m+2}})^{11}} \cdot \frac{1}{2^{m+2}} [4194304 \cdot \left(\frac{1}{2^{m+2}}\right)^{10} + \\ &+ 2134900736 \cdot \left(\frac{1}{2^{m+2}}\right)^9 + 37622906880 \cdot \left(\frac{1}{2^{m+2}}\right)^8 + 119325851648 \cdot \left(\frac{1}{2^{m+2}}\right)^7 + \\ &+ 159549259776 \cdot \left(\frac{1}{2^{m+2}}\right)^6 + 64406519868 \cdot \left(\frac{1}{2^{m+2}}\right)^5 + 9971828736 \cdot \left(\frac{1}{2^{m+2}}\right)^4 + \\ &+ 101038080 \cdot \left(\frac{1}{2^{m+2}}\right)^3 + 599696 \cdot \left(\frac{1}{2^{m+2}}\right)^2 + 32576 \cdot \frac{1}{2^{m+2}} + 4], \\ \sum_{n=0}^{\infty} n^{11} \cdot \frac{1}{2^{mn}} &= \frac{1}{(2^m - 1)^{12}} (2^m + 2036 \cdot 2^{2m} + 152637 \cdot 2^{3m} + 2203488 \cdot 2^{4m} + \\ &+ 9738114 \cdot 2^{5m} + 15724248 \cdot 2^{6m} + 9738114 \cdot 2^{7m} + 2203488 \cdot 2^{8m} + 152637 \cdot 2^{9m} + \\ &+ 2036 \cdot 2^{10m} + 2^{11m}). \end{aligned}$$

Namely there exists the sum identity

$$\begin{aligned} \frac{1^{11}}{2^m} + \frac{2^{11}}{2^{2m}} + \cdots + \frac{n^{11}}{2^{mn}} + \cdots &= \\ &= \frac{1}{(2^m - 1)^{12}} (2^m + 2036 \cdot 2^{2m} + 152637 \cdot 2^{3m} + 2203488 \cdot 2^{4m} + 9738114 \cdot 2^{5m} + \\ &+ 15724248 \cdot 2^{6m} + 9738114 \cdot 2^{7m} + 2203488 \cdot 2^{8m} + 152637 \cdot 2^{9m} + 2036 \cdot 2^{10m} + 2^{11m}), \\ m \in N_+. \end{aligned}$$

Conclusions:

- (1) base $2^m, 2^{2m}, 2^{3m}, 2^{4m}, 2^{5m}, 2^{6m}, 2^{7m}, 2^{8m}, 2^{9m}, 2^{10m}, 2^{11m}$
- (2) the sequence of coefficients is unimodal and symmetric ($a_k = a_{12-k}, 1 \leq k \leq 11$)
- (3) with one peak, $\frac{15724248}{(2^m - 1)^{12}}$ □

In particular, set $m=1$ in Theorem 1.22, the sum equality is derived as follows:

$$\begin{aligned} & \frac{1^{11}}{2} + \frac{2^{11}}{2^2} + \cdots + \frac{n^{11}}{2^n} + \cdots = \\ = & \frac{1}{(2^1 - 1)^{12}} (2^1 + 2036 \cdot 2^{2 \times 1} + 152637 \cdot 2^{3 \times 1} + 2203488 \cdot 2^{4 \times 1} + 9738114 \cdot 2^{5 \times 1} + \\ & + 15724248 \cdot 2^{6 \times 1} + 9738114 \cdot 2^{7 \times 1} + 2203488 \cdot 2^{8 \times 1} + 152637 \cdot 2^{9 \times 1} + \\ & + 2036 \cdot 2^{10 \times 1} + 2^{11 \times 1}) = 3245265146. \end{aligned}$$

3. Conclusions

By using the generating function method, it has been proven that the conjecture holds when $1 \leq r \leq 11$. For a given smaller r , we can prove this conjecture through calculation, but for a larger r , the calculation is very complex and it is difficult for us to prove this conjecture. For larger r , establishing an algorithm may prove this conjecture. Finally, it is fully proven that the difficulty coefficient of this conjecture is very high.

4. Conflict of interest

The authors declare no conflict of interest.

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References

1. Bondy, J A, Murty U S R: *Graph Theory with Applications*, Oxford: The Macmillan Press,Ltd, 1976.
2. June H.: Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs, *Journal of the American Mathematical Society*, **25**(3)(2012) 907-927.
3. Boros,G., Moll,V.: A criterion for unimodality, *The Electronic Journal of Combinatorics*,**6** (1999) 1-4.
4. Comtet, L.: *Advanced Combinatorics*, Netherlands:Springer Netherlands, 1974.
5. WANG ,T.M.:*Modern Combinatorics*, Dalian: Dalian University of Technology Press, 2008.
6. Richard P.S. *Enumerative Combinatorics*, London:Cambridge University Press, 1997.
7. Kheemeng, K., Engguan,T.: *Counting*, Signapore: World Scientific, 2013.
8. YANG, L.M., WANG, T.M.: The Genarating Function of Associted Numbers and The Representing Formula, *Int. J. Comb. Graph Theory Appl.*, **4**(2)(2019):71-93.

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